ON THE EFFICIENCY OF FINITE GROUPS

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BY

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Abstract

In Chapter 2 of this thesis we look at methods for finding efficient presentations of the transitive permutation groups of degree ≤ 12. Chapter 3 gives efficient presentations for certain direct products of groups including $PSL(2,\ p)^2$, $PSL(2,\ p) \times SL(2,\ 8)$, $PSL(2,\ p) \times C_2$, for prime $p \geq 5$ and $PSL(2,\ 25)^3$. Chapter 4 introduces a new class of inefficient groups and Chapter 5 gives a brief survey of some of the open problems relating to the efficiency of finite groups.

Declarations

I, Melanie Brookes, hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for a higher degree.

Signed **Date** 11/09/95

I was admitted to the Faculty of Science of the University of St Andrews under Ordinance General No. 12 on 01/10/92 and as a candidate for the degree of Ph.D. on 01/10/93.

Signec **Date** 11/09/95

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of Ph.D.

Signature of Superviso **Date** 11 Sept 1995

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Contents

Notation

Let A, B be sets:

$\mid A \mid$	the cardinality of A
$A \cap B$	the intersection of A and B
$A \cup B$	the union of A and B.

Let G, H be groups with $\mathbf{g}$, $\mathbf{g_i} \in \mathbf{G}$, $\mathbf{h} \in \mathbf{H}$:

$g \in G$	the element g is in G
$\mid G \mid$	the order of G
$Z(G)$	the centre of G
G'	the derived subgroup of G
$\mathrm{Aut}(G)$	the group of automorphisms of G
$G \cong H$	G is isomorphic to H
$G \leq H$	G is a subgroup of H
$G \lhd H$	G is a normal subgroup of H
$G \times H$	the direct product of G and H
G^n	the direct product of n copies of G
$G \otimes H$	the tensor product of G and H
G/H	the factor/quotient group of G by H

$[g, g_1]$	the commutator $g^{-1}g_1^{-1}gg_1$
$[G, H]$	the group generated by all elements of the form $g^{-1}h^{-1}gh$ where $G, H \leq L$ for some group L
$w(g_1, g_2, ..., g_n)$	a word in $g_1, g_2, ..., g_n$
$\langle g_1, g_2, ..., g_n \rangle$	the subgroup of G generated by $g_1, g_2, ..., g_n$.

Let θ be a homomorphism $\theta : \mathbf{G} \to \mathbf{H}$:

$\mathrm{Ker}(\theta)$	the kernel of θ in G
$\mathrm{Im}(\theta)$	the image of θ in H.

$\mathrm{hcf}(m, n)$	the highest common factor of the integers m and n
$\mathcal{Z}_n$	the set of integers modulo n
C_n	the cyclic group of order n
S_n	the symmetric group of degree n
A_n	the alternating group of degree n
$SL(m, n)$	the special linear group of $m \times m$ matrices with entries in the field of n elements
$PSL(m, n)$	the projective special linear group of $m \times m$ matrices with entries in the field of n elements
$PSU(m, n)$	the projective special unitary group of $m \times m$ matrices with entries in the field of n^2 elements.

Introduction

The efficiency of finite groups has been studied for many years. For example, see [2], [6], [17], [24], [28], [29]. In particular the efficiency of direct products has been the source of a great deal of study [1], [4], [7], [16]. In 1964 Swan [26] gave us the first known family of inefficient groups. Since then the gap between known efficient groups and known inefficient groups has been narrowing. The aim of this thesis is to narrow that margin slightly more.

Chapter 1 introduces the ideas fundamental to the study of efficiency and describes methods used in the work underlying the rest of the thesis. Chapter 2 describes methods useful in finding efficient presentations, in particular those for the transitive permutation groups of degree ≤ 12.

Chapter 3 gives some general results showing how to construct efficient presentations for direct products of groups which display certain properties. These results lead to efficient presentations for $PSL(2,\ p)^2$, $PSL(2,\ p) \times PSL(2,\ 25)$, $PSL(2,\ 27)^3$, $PSL(2,\ p) \times M_{11}$, $PSL(2,\ p) \times C_2$ where p is prime and ≥ 5 and many other examples are given.

In Chapter 4 we introduce a new family of inefficient groups and give a proof that they are inefficient along with several alternative representations for them. We also consider certain extensions of these groups.

Finally, Chapter 5 gives a brief survey of some of the open problems relating to efficiency.

Chapter 1. Some definitions and preliminary results

Section 1.1. Group presentations and deficiency

Let X be a set with $F = F(X)$ the free group on X, i.e. the group generated by X satisfying no relations other than those resulting from the group axioms. Every group G is a homomorphic image of a free group [19]. So, for any group G, $G = F/N$ for some free group F and normal subgroup N. Let R be a subset of F such that $N = \bar{R}$, the normal closure of R in F.

Definition. $\langle X \mid R \rangle$ is a *presentation* of G. We will often write $G = \langle X \mid R \rangle$, meaning that $\langle X \mid R \rangle$ is a presentation of G. G is *finitely presented* if there exists such a presentation for G with both X and R finite.

Example. A presentation for the cyclic group of order n is:

$$\langle a \mid a^n \rangle.$$

It is often more convenient to write these *relators* in R as *relations*, writing $r = 1$, $r \in R$, so that a presentation for the cyclic group of order n is:

$$\langle a \mid a^n = 1 \rangle,$$

and the dihedral group of order $2n$ has presentation:

$$\langle a,\ b \mid a^2 = b^n = (ab)^2 = 1\rangle.$$

If $\langle X \mid R\rangle$ is a presentation for the group G then the addition of the relation $r' = 1$ to the set $R = 1$ of relations yields a presentation for the factor group $G/\langle \bar{r'}\rangle$ where $\langle \bar{r'}\rangle$ is the normal closure of the relator r'.

Definition. For a finitely presented group, the *deficiency of the presentation* $\langle X \mid R\rangle$ is the integer $\mid R \mid - \mid X \mid$. The *deficiency of a group* G, $\mathrm{def}(G)$, is the minimum $\mid R \mid - \mid X \mid$ taken over all presentations $\langle X \mid R\rangle$ for G.

Example. The presentation for S_5 given below has deficiency two:

$$S_5 = \langle a,\ b \mid a^2 = b^5 = (ab)^4 = [a,\ b^2]^2 = 1\rangle.$$

A presentation for S_5 with deficiency one is:

$$S_5 = \langle x,\ y \mid x^{-2}y^4 = x^{-2}(xy)^5 = x^{-2}[x, y]^3 = 1\rangle.$$

It is often more convenient to write a relation $r = 1$ in the form $r_1 = r_2$ where $r = r_1 r_2^{-1}$. We can modify the last presentation this way to obtain:

$$S_5 = \langle x,\ y \mid x^2 = y^4 = (xy)^5 = [x,\ y]^3\rangle.$$

The deficiency of a finite group is always non-negative, although this is not a sufficient condition for the finiteness of a group.

In general, it is not an easy task to calculate the deficiency of a group. Indeed, the deficiencies of many groups are not known. We can obtain a lower bound for the deficiency of a group if we know its *Schur multiplier* which will be introduced in Section 1.4.

Section 1.2. Tietze transformations

Given two presentations, in general there is no way of deciding whether or not the two presentations define isomorphic groups. However, if we let G be a group with presentations $\langle X \mid R\rangle$ and $\langle Y \mid S\rangle$ we can pass from one presentation to the other by using a series of *Tietze transformations.* There are four of these, each operating by changing either the set of generators, X, of the presentation, or the set of relations, R, without affecting the group defined by the presentation. The four transformations are described below, each working from a given presentation $\langle X \mid R\rangle$ to give an equivalent presentation $\langle X' \mid R'\rangle$:

$T1$) If r is a word in the generators in X and the relation $r = 1$ holds in G then put $X' = X$, $R' = R \cup \{r\}$.

$T2$) If $r \in R$ is such that we have $r = 1$ in the group defined by $\langle X \mid R \setminus \{r\}\rangle$ then put $X' = X$, $R' = R \setminus \{r\}$.

$T3)$ If w is a word in X and z is a symbol not in X then

put $X' = X \cup \{z\}$, $R' = R \cup \{wz^{-1}\}$.

$T4)$ If $z \in X$ and w is a word in $X \setminus \{z\}$ such that $wz^{-1} \in R$

then substitute w for z in every other element of R

to give $\hat{R}$ and put $X' = X \setminus \{z\}$, $R' = \hat{R}$.

In short, these transformations allow us to add/remove redundant relations/generators.

Example. Consider the two presentations for S_5 given in Section 1.1:

$$P_1 = \langle a,\ b \mid a^2 = b^5 = (ab)^4 = [a,\ b^2]^2 = 1\rangle \text{ and}$$

$$P_2 = \langle x,\ y \mid x^2 = y^4 = (xy)^5 = [x,\ y]^3\rangle.$$

Let X be the set of generators and R the set of relations of presentation P_1. First apply $T3$ and add the generator $y = b^{-1}a$ so that we obtain the presentation:

$$\langle a,\ b,\ y \mid a^2 = b^5 = (ab)^4 = [a,\ b^2]^2 = b^{-1}ay^{-1} = 1\rangle.$$

The new relation gives us that $b = ay^{-1}$ and so we can apply $T4$ and substitute in ay^{-1} for b in the other relations to give:

$$\langle a,\ y \mid a^2 = (ay^{-1})^5 = (a^2y^{-1})^4 = [a,\ (ay^{-1})^2]^2 = 1\rangle.$$

Since $a^2 = 1$,

$$(ay^{-1})^5 = 1 \Leftrightarrow (ay^{-1})^{-5} = 1 \Leftrightarrow (ya)^5 = 1 \Leftrightarrow (ay)^5 = 1, \text{ and}$$

$$(a^2y^{-1})^4 = 1 \Leftrightarrow y^{-4} = 1 \Leftrightarrow y^4 = 1,$$

and so we can replace $(ay^{-1})^5 = (a^2y^{-1})^4 = 1$ by $y^4 = (ay)^5 = 1$. This replacement corresponds to two applications of $T1$ followed by two applications of $T2$. Now let $\langle X \mid R \rangle$ be the presentation:

$$\langle a,\ y \mid a^2 = y^4 = (ay)^5 = [a,\ (ay^{-1})^2]^2 = 1 \rangle.$$

It is easy to check, using **Cayley**, [8], or **GAP**, [22], that the relation $[a,\ y]^3 = 1$ holds in the group defined by this presentation and so we can apply $T1$ to give us the presentation $\langle X' \mid R' \rangle$ where $X' = X$ and $R' = R \cup \{[a,\ y]^3\}$. Again we can use the computer to show that, in this new presentation, the relation $[a,\ (ay^{-1})^2]^2 = 1$ is redundant and so we can apply $T2$ to give us the presentation:

$$\langle a,\ y \mid a^2 = y^4 = (ay)^5 = [a,\ y]^3 = 1 \rangle.$$

This presentation is obviously equivalent to the presentation:

$$\langle a,\ y \mid a^2 = a^{-2}y^4 = a^{-2}(ay)^5 = a^{-2}[a,\ y]^3 = 1 \rangle.$$

We can check by computer that the relation $a^2 = 1$ is redundant in this presentation, so applying $T2$ we are left with:

$$\langle a,\ y \mid a^2 = y^4 = (ay)^5 = [a,\ y]^3 \rangle.$$

Now we can apply $T3$, putting $x = a$ and then $T4$ to remove the redundant generator a, or simply replace the symbol a by the symbol x to obtain the presentation P_2 given earlier for S_5.

Section 1.3. Presentations of direct products and semidirect products

If, in general, we have two groups, $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$, then a presentation for their direct product is:

$$G \times H = \langle X,\ Y \mid R,\ S,\ \{[x,\ y] = 1 \mid x \in X,\ y \in Y\} \rangle.$$

Example. Consider the direct product of A_5 and C_2:

$$A_5 = \langle a,\ b \mid a^2 = b^5 = (ab)^3 = 1 \rangle,$$

$$C_2 = \langle x \mid x^2 = 1 \rangle, \text{ and}$$

$$A_5 \times C_2 = \langle a,\ b,\ x \mid a^2 = b^5 = (ab)^3 = x^2 = [a,\ x] = [b,\ x] = 1 \rangle.$$

We can also construct presentations of semidirect products of groups from the presentations of the groups themselves. A presentation for the semidirect product of H and G with respect to the homomorphism $\alpha : H \to \text{Aut } G$ is:

$$\langle X,\ Y \mid R,\ S,\ \{y^{-1}xy = x(y\alpha) \mid x \in X,\ y \in Y\} \rangle,$$

where $x(y\alpha)$ is the image of x under the automorphism $y\alpha$ of G, α having determined an action on G by H.

Example. C_2 acts on A_5 to give S_5:

$$S_5 = \langle a,\ b,\ x \mid a^2 = b^5 = (ab)^3 = x^2 = [a,\ x] = 1,\ x^{-1}bx = ab^{-1}abab^{-1} \rangle.$$

Section 1.4. The Schur multiplier and efficiency

Definition. If H is a group with a central subgroup Z and $G \cong H/Z$ then H is a *central extension* of G by Z.

Example 1.4.1. A central extension of A_5 by C_2 is defined by the presentation:

$$H = \langle a,\ b,\ z \mid a^2 = z,\ z^2 = b^5 = (ab)^3 = [a,\ z] = [b,\ z] = 1 \rangle.$$

Definition. If we have $G \cong H/Z$ and, in addition to $Z \leq Z(H)$, we have $Z \leq H'$ then H is a *stem extension* of G by Z.

The central extension of A_5 given by the presentation above also turns out to be a stem extension. This can be verified using the ideas introduced in Section 1.7.

Definition. For G a finite group, the stem extensions of highest order are called *covering groups* for G.

In general, a covering group for a group G is not unique. However, in the case where G is perfect, it is unique. See [25] for a proof.

In fact, the given central extension of A_5 by C_2 is a covering group for A_5. Since A_5 is perfect, it is unique.

Definition 1.4.2. If G is a finite group and H has maximum order subject to the conditions:

1) $H/A \cong G$, where

2) $A \leq Z(H) \cap H'$

(i.e. H is a covering group for G) then the group A is called the *Schur multiplier* or just the *multiplier* of G, $M(G)$. The multiplier of a group is unique [23].

So, the Schur multiplier of A_5 is C_2.

The multiplier was introduced by I. Schur in [23]. His definition was in terms of projective representations. In that paper he proves that the multiplier is always finite and gives a bound for its order.

Since $M(G)$ is a finite abelian group we can write it as the direct product of cyclic groups. The minimum number of generators required for the multiplier, $M(G)$, is called the *rank* of the multplier, rank$(M(G))$.

In [13], Hopf provides us with another definition of the multiplier: If G is finite and $G = \langle X \mid R \rangle$ i.e. $G = F/\bar{R}$ where $F = F(X)$ is the free group on the symbols of X, $R \subseteq F$ and $\bar{R}$ is the normal closure of R in F then:

$$M(G) \cong \frac{F' \cap \bar{R}}{[F,\ \bar{R}]}.$$

Looking at the multiplier this way leads naturally to the result (Lemma 1.2 in [30]):

Theorem 1.4.3. *If G is a finite group with deficiency d then:*

$$\operatorname{rank}(M(G)) \leq d \quad .$$

A proof of this is given in [30], although the result is originally due to Schur.

Definition. If we have equality in the inequality of Theorem 1.4.3, then G is said to be *efficient.* If there is strict inequality then G is said to be *inefficient.*

Many classes of groups are known to be efficient e.g. all abelian groups, all metacyclic groups (see [15] for a proof) and many of the simple groups. A_5 is efficient since the multiplier is of rank one and there exists a presentation with two generators and three relations. There are also known examples of inefficient groups. These are given in Chapter 4. However the question of efficiency remains unanswered for p-groups, some simple groups and a variety of conjectures and difficult unsolved problems have arisen in this area. Some of these are given in Chapter 5.

Section 1.5. Some useful results on the multiplier

Lemma 1.5.1. (Lemma 4.1 in [30]) *Let H be a group and let $Z \leq Z(H)$ have finite index. If G is the factor group $G = H/Z$ then $H' \cap Z$ is an epimorphic image of $M(G)$.*

So, if H is perfect, $Z \leq Z(H)$ has finite index in H and $G = H/Z$, then Z is an epimorphic image of $M(G)$.

Lemma 1.5.2. (Corollary 4.3 in [30]) *The exponent (i.e. the lowest common multiple of the orders of all the elements) of $M(G)$ divides the index of every cyclic subgroup of G. In particular, it divides $\mid G \mid$.*

Lemma 1.5.3. (Corollary 4.4 in [30]) *$M(G)_p \leq M(G_p)$ where K_p denotes a Sylow p-subgroup of the group K.*

Theorem 1.5.4. (Theorem 5.1 in [30]) *For any two finite groups, A and B:*

$$M(A \times B) \cong M(A) \times M(B) \times (A \otimes B).$$

$A \otimes B$ is the *tensor product* of A and B. In order to use the tensor product to calculate the multplier of a direct product using Theorem 1.5.4 we need only use the following properties of the tensor product:

1) $A \otimes B = B \otimes A$,

2) $A \otimes B = A/A' \otimes B/B'$,

3) $C_n \otimes C_m = C_{\mathrm{hcf}(m,\, n)}$,

4) $A \otimes (B_1 \times B_2) = (A \otimes B_1) \times (A \otimes B_2)$,

where A, B, B_1, B_2 are any finite groups. Since $A \otimes B = A/A' \otimes B/B'$ then, for perfect groups, the multiplier of the direct product is isomorphic to the direct product of the multipliers, $M(A \times B) \cong M(A) \times M(B)$.

Section 1.6. The Todd-Coxeter coset enumeration process and the Reidemeister-Schreier rewriting process

The Todd-Coxeter coset enumeration process was introduced in the important paper [27]. From a finite presentation of a finite group G it gives us the order of the group and a permutation representation for G. When modified to enumerate the cosets of a subgroup of G in G (given by the subgroup generators which are words in the group generators) it gives us the index of the subgroup, a permutation representation of G on the (right) cosets of the subgroup and leads us to the Reidemeister-Schreier process for finding a presentation of this subgroup.

Let $G = \langle X \mid R \rangle$. For each relation $r \in R$, $r = x_1 x_2 x_3 \ldots x_n$, where each x_i is in X or is the inverse of some element of X, we draw up a table having $n+1$ columns and put a 1 in the first and last places of the first row:

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & \ldots\ldots & x_n & \\ 1 & & & & & & 1\,. \end{array}$$

Now we choose an empty space next to a 1 in one of the tables, put a 2 in it and create a second row:

$$\begin{array}{ccccccc} & x_1 & & x_2 & x_3 & \ldots\ldots & x_n & \\ 1 & & 2 & & & & & 1 \\ 2 & & & & & & & 2\,. \end{array}$$

We record the definition $1x_1 = 2 \Leftrightarrow 2x_1^{-1} = 1$ in a monitor table with columns corresponding to all the elements in X and rows corresponding to the symbols in the relator tables. We now put a 2 in any space to the right of any 1 with an x_1 between, or to the left of, any 1 with an x_1^{-1} between and a 1 to the left of any 2 with an x_1 between, or to the right of, any 2 with an x_1^{-1} between. This process is known as *scanning*. We now introduce the symbol 3 by placing it in an empty space in the table adjacent to some filled space, say $2x_2 = 3$, record the definition, and introduce a third row to the tables. We now begin the scanning process again, putting a 3 to the right of every 2 with an x_2 between and so on. Now we introduce the symbol 4 adjacent to some filled space, add a fourth row to the table and scan once more. We continue in this way until we define some j, say $ix_q = j$ which completes a row:

$$\begin{array}{ccccccc} \cdots & & x_q & & x_{q+1} & & \cdots \\ & i & & & & k & \end{array} \longrightarrow \begin{array}{ccccccc} \cdots & & x_q & & x_{q+1} & & \cdots \\ & i & & j & & k & \end{array}.$$

This definition gives us the 'bonus' information that $jx_{q+1} = k$. Through scanning, this may in turn lead to further pieces of 'bonus' information. We also store this bonus information in the monitor table. We carry on in this way until the tables are complete.

Example. Enumerate $S_3 = \langle a,\ b \mid a^2 = b^3 = (ab)^2 = 1\rangle$. We obtain relator tables:

$$\begin{array}{ccccc} & a & & a & \\ 1 & \bullet & 2 & \star & 1 \\ 2 & & 1 & & 2 \\ 3 & \bullet & 4 & \star & 3 \\ 4 & & 3 & & 4 \\ 5 & \bullet & 6 & \star & 5 \\ 6 & & 5 & & 6 \end{array} \qquad \begin{array}{ccccccc} & b & & b & & b & \\ 1 & \bullet & 3 & \bullet & 5 & \star & 1 \\ 2 & & 6 & \star & 4 & & 2 \\ 3 & & 5 & & 1 & & 3 \\ 4 & & 2 & & 6 & & 4 \\ 5 & & 1 & & 3 & & 5 \\ 6 & & 4 & & 2 & & 6 \end{array}$$

	a		b		a		b
1	2	$\star$	6	$\cdot$	5		1
2	1		3		4	$\star$	2
3	4		2		1		3
4	3		5		6		4
6	5		1		2		6

where $\bullet$ indicates a definition and $\star$ indicates a bonus piece of information. The monitor table for this enumeration is:

	a	b
1	2	3
2	1	6
3	4	5
4	3	2
5	6	1
6	5	4 .

So, unsurprisingly, we find that S_3 has order six. The monitor table gives us permutation generators for S_3. The generator a maps symbol 1 to symbol 2, symbol 2 to symbol 1, symbol 3 to 4 and so on to give us that $a = (1\ 2)(3\ 4)(5\ 6)$ and $b = (1\ 3\ 5)(2\ 6\ 4)$.

It may occur during the scanning process that we appear to get a contradiction. For example, we may obtain the bonus information that $ix = k$ when we already have that $ix = j$, $j \neq k$. This is called *coset coincidence* and in this case, to avoid contradiction, we must have $j = k$ so choose the smaller symbol and relabel the larger one throughout. This may require the renaming of many more of the symbols in order to have a consecutive set of numbers to deal with. In such cases, once these modifications have been made, the enumeration can continue as before. The use of this modification means that this method is very forgiving if the scanning process is not completed at any stage.

We can use a slight modification of this method of enumeration to enumerate the cosets of a subgroup H of a a finite group G. This gives us the index of H in G and a permutation representation of G on the (right) cosets of H. Again we have tables corresponding to the relators of G. This time we also have a table for each generator of H, constructed in the same way, but only having one row, beginning and ending with a 1.

Example 1.6.1. Consider the subgroup $H = \langle b,\ ab^2ab^{-2}ab^2\rangle$ of $A_5 = \langle a,\ b \mid a^2 = b^5 = (ab)^3 = 1\rangle$. With $\bullet$ representing a definition and $\star$ representating a bonus piece of information we obtain the following relator tables:

$$\begin{array}{ccccc}
 & a & & a & \\
1 & \bullet & 2 & \star & 1 \\
2 & & 1 & & 2 \\
3 & \bullet & 4 & \star & 3 \\
4 & & 3 & & 4 \\
5 & & 5 & & 5 \\
6 & & 6 & & 6
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & b & & b & & b & & b & & b & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 3 & & 5 & \bullet & 6 & \star & 4 & & 2 \\
3 & & 5 & & 6 & & 4 & & 2 & & 3 \\
4 & & 2 & & 3 & & 5 & & 6 & & 4 \\
5 & & 6 & & 4 & & 2 & & 3 & & 5 \\
6 & & 4 & & 2 & & 3 & & 5 & & 6
\end{array}$$

$$\begin{array}{ccccccccccccc}
 & a & & b & & a & & b & & a & & b & \\
1 & & 2 & & 3 & & 4 & \star & 2 & & 1 & & 1 \\
2 & & 1 & & 1 & & 2 & & 3 & & 4 & & 2 \\
3 & & 4 & & 2 & & 1 & & 1 & & 2 & & 3 \\
4 & & 3 & & 5 & & 5 & & 6 & \star & 6 & & 4 \\
5 & & 5 & & 6 & & 6 & & 4 & & 3 & & 5 \\
6 & & 6 & & 4 & & 3 & & 5 & & 5 & & 6
\end{array},$$

and the following tables corresponding to the subgroup generators:

$$\begin{array}{ccc}
 & b & \\
1 & & 1
\end{array}$$

$$\begin{array}{ccccccccccccccccccc}
 & a & & b & & b & & a & & b^{-1} & & b^{-1} & & a & & b & & b & \\
1 & & 2 & \bullet & 3 & \bullet & 5 & \star & 5 & & 3 & & 2 & & 1 & & 1 & & 1
\end{array}.$$

The monitor table for this enumeration is then:

	a	b
1	2	1
2	1	3
3	4	5
4	3	2
5	5	6
6	6	4 .

So, H has index 6 in G. The generators for the permutation representation for A_5 on the (right) cosets of H are $(1\ 2)(3\ 4)$ and $(2\ 3\ 5\ 6\ 4)$. We shall return to this example later in this section.

Returning to the general case, consider a finite group G with a subgroup H of index n in G so that the cosets of H in G can be written Hg_1, Hg_2, ..., Hg_n for some $g_1, g_2, ..., g_n \in G$. For each $x \in G$ we can define a permutation on n symbols as follows. Define ϕ_x so that $Hg_ix = Hg_{\phi_x(i)}$. Now define a homomorphism $\theta : G \to S_n$, $\theta(x) = \phi_x$. Now $G/\mathrm{Ker}(\theta) \cong \mathrm{Im}(\theta) \leq S_n$ so, if $\mathrm{Ker}(\theta)$ is trivial, we have $G \cong \mathrm{Im}(\theta) \leq S_n$. In this case coset enumeration of G over H will yield a permutation representation for G. So, since $\mathrm{Ker}(\theta) \lhd G$ and $\mathrm{Ker}(\theta) \leq H$, if H contains no subgroups normal in G then coset enumeration of G over H will yield a permutation representation of G. In particular, if G is simple, coset enumeration over any subgroup will yield a permutation representation for G.

So, Example 1.6.1 gives us that A_5 is generated by (1 2)(3 4) and (2 3 5 6 4).

Todd-Coxeter coset enumeration over a subgroup of finite index always converges, [21], although, in general, there is no way of knowing in advance how many symbols will be needed.

The coset monitor table in Example 1.6.1 can be used to obtain a presentation for the subgroup H. The method is known as the *Reidemeister-Schreier rewriting process*, [20].

Working from the coset monitor table, we begin by identifying the entries corresponding to definitions in the coset enumeration. Now we attach a y_i to the left of each of the other entries. These y_i are the generators for our presentation of H. Since there are gk entries in the coset table, where $g = \mid X \mid$ and $k = \mid G : H \mid$, there are $(g-1)k+1$ of these. The rk relations we obtain, where $r = \mid R \mid$, come from the rk relations of the form $i = ir_j$, $1 \leq i \leq k$, $r_j \in R$. We obtain the relations for H as follows. Consider some $i = ir_j = ix_1x_2x_3...x_n$. The modified coset table gives us that $ix_1 = y_lm$ (or $ix_1 = m$ if this was a definition in the coset enumeration). We can say then that $i = y_lmx_2x_3...x_n$. Now look at what mx_2 gives us in the modified coset table. Continuing in this way we obtain $i = y_ly_qy_s...y_ti$ for some $1 \leq l, q, s, ..., t \leq (g-1)k+1$ and we have $y_ly_qy_s...y_t = 1$ in H. The method is best illustrated by example.

Example. Continuing where we left off with the enumeration of

$A_5 = \langle a,\ b \mid a^2 = b^5 = (ab)^3 = 1\rangle$ over $H = \langle b,\ ab^2ab^{-2}ab^2\rangle$.

The modified coset table is:

	a	b
1	2	$y_1 1$
2	$y_2 1$	3
3	4	5
4	$y_3 3$	$y_4 2$
5	$y_5 5$	6
6	$y_6 6$	$y_7 4$

where $y_i,\ 1 \le i \le 7$ will be the generators in our presentation for H. There will be eighteen relations, six corresponding to each of the three relations in the presentation for A_5. Firstly, consider the six relations corresponding to $a^2 = 1$:

$$1a^2 = 1 \Leftrightarrow 2a = y_2 1 = 1 \Leftrightarrow y_2 = 1,$$

$$2a^2 = 2 \Leftrightarrow y_2 1a = y_2 2 = 2 \Leftrightarrow y_2 = 1.$$

Similarly, $3a^2 = 3$ and $4a^2 = 4$ give us that $y_3 = 1$.

$$5a^2 = 5 \Leftrightarrow y_5 5a = y_5^2 5 = 5 \Leftrightarrow y_5^2 = 1.$$

Similarly, $6a^2 = 6$ gives us that $y_6^2 = 1$. Now consider the six relations corresponding to $b^5 = 1$:

$$1b^5 = y_1 1b^4 = y_1^2 1b^3 = y_1^3 1b^2 = y_1^4 1b = y_1^5 1 = 1 \Leftrightarrow y_1^5 = 1,$$

$$2b^5 = 3b^4 = 5b^3 = 6b^2 = y_7 4b = y_7 y_4 2 = 2 \Leftrightarrow y_7 y_4 = 1,$$

$$3b^5 = 5b^4 = 6b^3 = y_7 4b^2 = y_7 y_4 2b = y_7 y_4 3 = 3 \Leftrightarrow y_7 y_4 = 1,$$

$$4b^5 = y_4 2b^4 = y_4 3b^3 = y_4 5b^2 = y_4 6b = y_4 y_7 4 = 4 \Leftrightarrow y_4 y_7 = 1,$$

$$5b^5 = 6b^4 = y_7 4b^3 = y_7 y_4 2b^2 = y_7 y_4 3b = y_7 y_4 5 = 5 \Leftrightarrow y_7 y_4 = 1,$$

$$6b^5 = y_7 4b^4 = y_7 y_4 2b^3 = y_7 y_4 3b^2 = y_7 y_4 5b = y_7 y_4 6 = 6 \Leftrightarrow y_7 y_4 = 1.$$

Finally, consider the six relations derived from $(ab)^6 = 1$:

$$1(ab)^3 = 2babab = 3abab = 4bab = y_4 2ab = y_4 y_2 1b = y_4 y_2 y_1 1 = 1$$

$$\Leftrightarrow y_4 y_2 y_1 = 1,$$

$$2(ab)^3 = y_2 1babab = y_2 y_1 1abab = y_2 y_1 2bab = y_2 y_1 3ab$$

$$= y_2 y_1 4b = y_2 y_1 y_4 2 = 2 \Leftrightarrow y_2 y_1 y_4 = 1,$$

$$3(ab)^3 = 4babab = y_4 2abab = y_4 y_2 1bab = y_4 y_2 y_1 1ab$$

$$= y_4 y_2 y_1 2b = y_4 y_2 y_1 3 = 3 \Leftrightarrow y_4 y_2 y_1 = 1,$$

$$4(ab)^3 = y_3 3babab = y_3 5abab = y_3 5bab = y_3 y_5 6ab$$

$$= y_3 y_5 y_6 6b = y_3 y_5 y_6 y_7 4 = 4 \Leftrightarrow y_3 y_5 y_6 y_7 = 1,$$

$$5(ab)^3 = y_5 5babab = y_5 6abab = y_5 y_6 6bab = y_5 y_6 y_7 4ab$$

$$= y_5 y_6 y_7 y_3 3b = y_5 y_6 y_7 y_3 5 = 5 \Leftrightarrow y_5 y_6 y_7 y_3 = 1,$$

$$6(ab)^3 = y_6 6babab = y_6 y_7 4abab = y_6 y_7 y_3 3bab = y_6 y_7 y_3 5ab$$

$$= y_6y_7y_3y_55b = y_6y_7y_3y_56 = 6 \Leftrightarrow y_6y_7y_3y_5 = 1.$$

So, a presentation for H is:

$$H = \langle y_1,\ y_4,\ y_5,\ y_6,\ y_7 \mid y_1^5 = y_5^2 = y_6^2 = y_1y_4$$

$$= y_4y_7 = y_5y_6y_7 = 1\rangle.$$

We can now use Tietze transformations to reduce this presentation to a more palatable form. We can use the fifth and sixth relations to eliminate the redundant generators y_4 and y_7 to give us:

$$H = \langle y_1,\ y_5,\ y_6 \mid y_1^5 = y_5^2 = y_6^2 = y_5y_6y_1^{-1} = 1\rangle.$$

Now we can eliminate the redundant generator y_1 using the forth relation to give:

$$H = \langle y_5,\ y_6 \mid y_5^2 = y_6^2 = (y_5y_6)^5 = 1\rangle.$$

This is a standard presentation for the dihedral group of degree 5. Hence G has order 60.

Consider a finite group $G = \langle X \mid R\rangle$ with a subgroup H of index k. Let $\mid X \mid = g$, $\mid R \mid = r$. Coset enumeration of G over H yields a table with k rows and g columns, hence gk entries. Since $k-1$ of these correspond to definitions made in the enumeration the number of generators in the presentation that the Reidemeister-Schreier process generates is $gk - k + 1 = k(g-1) + 1$. The number of relations in the presentation generated for the subgroup is rk. So, if the deficiency of the presentation for G is $d = r - g$ then the deficiency of the presentation of the subgroup will be $rk - gk + k - 1 = k(d + 1) - 1$. This line of thinking provides us with a useful tool in proving that some groups are inefficient. This will be described in Chapter 4.

Although we know that coset enumeration will always terminate for a finite group it is obviously not practical to carry out the enumeration by hand for the vast majority of groups under investigation. There are several computer implementations of this process. See [8], [11], [12], [22].

Section 1.7. Abelianisation of a group

For $G = \langle X \mid R \rangle$ a group, the derived group of G, G' is the normal closure of the set of words of the form $x_1^{-1}x_2^{-1}x_1x_2$, $x_1,\ x_2 \in X$. The factor group:

$$G/G' = \langle X \mid R,\ \{x_1^{-1}x_2^{-1}x_1x_2 \mid x_1,\ x_2 \in X\}\rangle$$

is obviously the largest abelian factor group of G and is known as the *abelianisation* of G.

G/G' can be represented by a matrix, each column corresponding to one of the generators in X and each row to a relator in R. Since G/G' is abelian we can gather together generators in each relator in R so that the relators are of the form $r_i = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ... x_n^{\alpha_n}$ where $x_j \neq x_k$ for any $1 \leq j,\ k \leq n$. Then we say α_j is the *exponent sum* of x_j in r_i. So, in the matrix, position $(i,\ j)$ is filled by the exponent sum of x_j in relator r_i. Now, Tietze transformations involving the relators of G/G' correspond to row operations on this matrix and those involving the generators of G/G' correspond to column operations. Of course, only row and column operations involving integral multiples of rows and columns correspond to valid Tietze transformations.

We can most easily obtain information about G/G', and so in turn G', if we reduce this matrix, via a series of row and column operations, to a matrix such that the only non-zero entries are in positions $(i,\ i)$. Since G/G' is abelian, it is isomorphic to a direct product of cyclic groups. This direct product can be written down directly from this form of the relation matrix. The best way to illustrate this is by example. When we write:

$$A \quad \underset{prow(c)+qrow(d)}{\overset{mrow(a)+nrow(b)}{\longrightarrow}} \quad B$$

we will mean that in matrix A we first replace the row a by $m \times$ row $a + n \times$ row b. Then, in resulting matrix, we replace row c by $p \times$ row $c + q \times$ row d to obtain the matrix B. Only when we have $m,\ p = \pm 1$ is it obvious that such row operations correspond to valid Tietze transformations.

Example. $S_3 = \langle a,\ b \mid a^2 = b^3 = (ab)^2 = 1\rangle$ and so $S_3/S_3' = \langle a,\ b \mid a^2 = b^3 = a^2b^2 = [a,\ b] = 1\rangle$. Consider the three by two matrix with columns 1 and 2 corresponding to $a,\ b$ respectively and rows 1, 2 and 3 corresponding to the relators $a^2,\ b^3,\ a^2b^2$ respectively:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 2 & 2 \end{pmatrix} \quad \begin{matrix} row(3) - row(1) \\ row(2) - row(3) \\ \longrightarrow \\ row(3) - 2row(2) \end{matrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So, the abelianisation of S_3 also has a presentation $\langle a,\ b \mid a^2 = b = [a,\ b] = 1\rangle$ which is obviously a presentation for $C_2 \times C_1 \cong C_2$.

We can use these ideas to see that a group presentation with negative deficiency defines an infinite group. Consider such a presentation. Let $G = \langle X \mid R\rangle$ where $\mid X \mid = n,\ \mid R \mid = m,\ n > m$. Representing the abelianisation of G by a matrix as described, and using row and column operations to get it in the required form we obtain, generally, a matrix:

$$\begin{pmatrix} \beta_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \beta_3 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & & . & . & . & & . & . \\ . & . & . & & . & . & . & & . & . \\ . & . & . & & . & . & . & & . & . \\ 0 & 0 & 0 & \dots & \beta_{m-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \beta_m & 0 & \dots & 0 & 0 \end{pmatrix}.$$

So, a presentation for the abelianisation of G is:

$$G/G' = \langle x_1,\ x_2,\ \dots,\ x_n \mid x_1^{\beta_1} = x_2^{\beta_2} = \dots = x_m^{\beta_m} = 1,$$
$$\{x_j^{-1}x_k^{-1}x_jx_k = 1 \mid 1 \leq j,\ k \leq n\}\rangle$$

which is a presentation for $C_{\beta_1} \times C_{\beta_2} \times \dots \times C_{\beta_m} \times (C_\infty)^{n-m}$. Since now G' has infinite index in G then G must be infinite. Of course G is infinite if any of the β_i are zero, whether or not the deficiency is negative and G may well be infinite even if its deficiency is positive and the abelianisation finite if G' is infinite.

However, in this thesis we only deal with finite groups. We use these ideas to obtain information about the derived groups of groups and also to show that certain groups are perfect.

Consider again the central extension of A_5 given in Example 1.4.1. We can use this matrix method to show that this is a stem extension. We have:

$$H = \langle a,\ b,\ z \mid a^2 = z,\ z^2 = b^5 = (ab)^3 = [a,\ z] = [b,\ z] = 1\rangle.$$

Let columns 1, 2 and 3 represent a, b and z respectively. Let rows 1 to 4 represent relators $(ab)^3$, b^5, a^2z^{-1} and z^2 respectively. Then we have:

$$\begin{pmatrix} 3 & 3 & 0 \\ 0 & 5 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[row(3) - 2row(1)]{row(1) - row(3)} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ 0 & -6 & -3 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ 0 & -6 & -3 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[row(3) + 6row(2)]{-row(2) - row(3)} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 15 \\ 0 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 15 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{row(3) - 7row(4)} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Now we can get the required form by removing any unwanted non-zero entries in column 3 using row 3, then any unwanted non-zero entries in column 2 using row 2. So we obtain:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since we have only performed row operations to obtain this matrix, the columns represent the original generators and so we can see that z is trivial in H/H'. So z is in H' and we have that H is a stem extension of A_5. We have also shown here that H is perfect since H/H' is trivial.

It is perhaps worth noting that, since we could have written down the relations of the group in any order, any permutation of the rows in the relation matrix has no effect on the abelian group it represents. Therefore, we do not necessarily require the non-zero entries in positions $(i,\ i)$ to determine G/G'. Any permutation of the rows of a matrix in this form will give us exactly the same information.

Chapter 2. Finding efficient presentations

Section 2.1. Transitive permutation groups

In this chapter we describe methods useful in finding efficient presentations using transitive permutation groups as examples. We can study the efficiency of the transitive permutation groups of degree ≤ 12 systematically with the help of **Cayley**. **Cayley**, [8], has library modules corresponding to all the transitive permutation groups of degree ≤ 12 which give the order of the groups, permutation generators and the block systems of the groups. The methods described in this chapter lead quite easily to efficient presentations for all transitive permutation groups of degree ≤ 6. Craig Wotherspoon [31] has found efficient presentations for most of these transitive permutation groups of degrees 7, 8, 10 and 11. I have looked at some of the remaining cases and those of degree 9. The only transitive permutation groups of degree ≤ 11 for which we do not now know the deficiency are A_n, S_n, $n \geq 9$.

Definition. A permutation group G of degree n is *transitive* if, for every two symbols α, β, $1 \leq \alpha,\ \beta \leq n$, $\exists\, g \in G$ such that $\alpha g = \beta$, that is G has exactly one orbit.

Section 2.2. Finding the Schur multiplier

In order to find an efficient presentation for a group G, we first need to know its Schur multiplier. In general this is a difficult problem. However, if G is not too large, we can make use of the *darstellungsgruppe* function in **Cayley** to obtain a covering group for G and hence the multiplier. For a permutation group G, the following **Cayley** procedure gives a presentation for G, a covering group H for G, and only fails to give the multiplier in the case when the multiplier is not of square free order, is not elementary abelian and is a proper subgroup of $Z(H) \cap H'$. In such a case we can easily work out what the multiplier is by looking at the orders of the 'extra' generators of H.

```
procedure schurmultiplier(g);


r=relations(g);
h=darstell(g);
print '';
print g; print 'order ';print order(g);
print r;
print composition factors(g);
print '';
```

```
print 'Covering group of g:';
print h; ; print '';print 'order ';print order(h);
print 'The Schur multiplier must have order';
m=order(h)/order(g);
print m;
n=0;
for i=2 to order(g) do
  if (m mod i^2) eq 0 then
    n=n+1;
  end;
end;
if n eq 0 then
  if m eq 1 then
    print 'so is trivially of rank zero.';
    else print 'so is the cyclic group of this order';
    print 'and has rank one.';
  end;
  else d=derived group(h);
  z=center(h);
  n=z meet d;
```

```
    if cyclic(n) then
     print 'and is the cyclic group of this order';
     print 'and has rank one.';
     else if elementary abelian(n) then
      print 'and is elementary abelian.';
      else if order(n) eq m then
       print 'and is the group defined by';
       print relations(n);
       else print 'and requires further analysis.';
      end;
     end;
    end;
  end;
  print '';
  print '************************************';
```

In fact, this procedure gives us the Schur multiplier for all the transitive permutation groups of degree up to and including twelve. Some examples of the output are as follows:

GROUP G OF ORDER 54 = 2 * 3^3 IS A SUBGROUP

GENERATOR:

A = (1,6,2,4,3,5)(8,9)

B = (1,7)(2,8)(3,9)(4,6)

order

54

[B^2, A^6, A^3 B A^ − 1 B A B]

COMPOSITION FACTORS OF GROUP G

```
  G
  |   Cyclic(2)
  *
  |   Cyclic(2)
  *
  |   Cyclic(3)
  *
  |   Cyclic(3)
  1
```

Covering group of g:

GROUP H

RELATORS :

H.2^2

H.1^6 H.3

H.1^3 H.2 H.1^-1 H.2 H.1 H.2

H.1^-1 H.3^-1 H.1 H.3

H.2^-1 H.3^-1 H.2 H.3

order

54

The Schur multiplier must have order

1

so is trivially of rank zero.

****

GROUP G OF ORDER 5040 = 2^4 * 3^2 * 5 * 7 IS A SUBGROUP

GENERATORS :

A = (1,2,3,4,5,6,7)

B = (1,2)

order

5040

[B^2, A^7, (A^2 B A^ − 2 B)^2, (A B A^ − 1 B)^3, (A B^ − 1)^6]

COMPOSITION FACTORS OF GROUP G

```
  G

  |   Cyclic(2)

  *

  |   Alternating(7)

  1

Covering group of g:


GROUP H

RELATORS :

H.2^2

H.1^7 H.5^-1

H.1^2 H.2 H.1^-2 H.2 H.1^2 H.2 H.1^-2 H.2 H.3^-1

H.1 H.2 H.1^-1 H.2 H.1 H.2 H.1^-1 H.2 H.1 H.2 H.1^-1 H.2 H.4^-1

H.1 H.2^-1 H.1 H.2^-1 H.1 H.2^-1 H.1 H.2^-1 H.1 H.2^-1 H.1 H.2^-1 H.5^-1

H.1^-1 H.3^-1 H.1 H.3

H.2^-1 H.3^-1 H.2 H.3

H.1^-1 H.4^-1 H.1 H.4

H.2^-1 H.4^-1 H.2 H.4

H.3^-1 H.4^-1 H.3 H.4

H.1^-1 H.5^-1 H.1 H.5
```

```
H.2^-1 H.5^-1 H.2 H.5

H.3^-1 H.5^-1 H.3 H.5

H.4^-1 H.5^-1 H.4 H.5

order

10080

The Schur multiplier must have order

2

so is the cyclic group of this order

and has rank one.

***********************************

GROUP G OF ORDER 24 = 2^3 * 3 IS A SUBGROUP

GENERATORS :

A = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)

B = (1,3)(2,4)(5,11)(6,12)(7,9)(8,10)

C = (1,5)(2,6)(3,7)(4,8)

order

24

[ A^2, B^2, C^2, (A B)^2, (A C)^2, (B C)^6 ]

COMPOSITION FACTORS OF GROUP G

------------------------------
```

```
  G
  |   Cyclic(2)
  *
  |   Cyclic(2)
  *
  |   Cyclic(2)
  *
  |   Cyclic(3)
  1
Covering group of g:
GROUP H
RELATORS :
H.1^2
H.2^2 H.4
H.3^2 H.5
(H.1 H.2)^2
(H.1 H.3)^2
H.2 H.3 H.2 H.3 H.2 H.3 H.2 H.3 H.2 H.3 H.2 H.3 H.6^-1
H.1^-1 H.4^-1 H.1 H.4
H.2^-1 H.4^-1 H.2 H.4
H.3^-1 H.4^-1 H.3 H.4
```

```
H.1^-1 H.5^-1 H.1 H.5

H.2^-1 H.5^-1 H.2 H.5

H.3^-1 H.5^-1 H.3 H.5

H.4^-1 H.5^-1 H.4 H.5

H.1^-1 H.6^-1 H.1 H.6

H.2^-1 H.6^-1 H.2 H.6

H.3^-1 H.6^-1 H.3 H.6

H.4^-1 H.6^-1 H.4 H.6

H.5^-1 H.6^-1 H.5 H.6

order

192

The Schur multiplier must have order

8

and is elementary abelian.

**********************************

GROUP G OF ORDER 144 = 2^4 * 3^2 IS A SUBGROUP

GENERATORS :

A = (1,10,5,11)(2,9,6,12)(3,7,4,8)

B = (1,8,3,9)(2,7,4,10)(5,12)(6,11)

order

144
```

```
[ A^4, (A B^ - 1)^2, B^4, (A B)^6 ]

COMPOSITION FACTORS OF GROUP G

------------------------------

  G

  |   Cyclic(2)

  *

  |   Cyclic(2)

  *

  |   Cyclic(2)

  *

  |   Cyclic(2)

  *

  |   Cyclic(3)

  *

  |   Cyclic(3)

  1

Covering group of g:

GROUP H

RELATORS :

H.1^4 H.3^-1

(H.1 H.2^-1)^2
```

```
H.2^4

H.1 H.2 H.1 H.2 H.1 H.2 H.1 H.2 H.1 H.2 H.1 H.2 H.4^-1

H.1^-1 H.3^-1 H.1 H.3

H.2^-1 H.3^-1 H.2 H.3

H.1^-1 H.4^-1 H.1 H.4

H.2^-1 H.4^-1 H.2 H.4

H.3^-1 H.4^-1 H.3 H.4

order

1728

The Schur multiplier must have order

12

and is the group defined by

[ N.2^2, N.1 N.2 N.1^ - 1 N.2, N.1^6 ]

**********************************
```

Using this information, we can investigate the efficiency of these transitive permutation groups. If a group is cyclic or metacyclic we know that it is efficient, [15]. Otherwise we must find an efficient presentation or show that none exists. In finding an efficient presentation, we apply Tietze transformations to the presentation given by *schurmultiplier*. We can use any computer implementation of coset enumeration to see whether or not different manipulations of this presentation correspond to valid Tietze transformations.

Example. The procedure *schurmultiplier* gives us that the group $t9n13$ in the **Cayley** library has presentation:

$$t9n13 = \langle a,\ b \mid a^6 = b^2 = a^3ba^{-1}bab = 1\rangle,$$

and trivial Schur multiplier. An efficient presentation for this group is:

$$t9n13 = \langle a,\ b \mid a^6 = a^3ba^{-1}bab^{-1} = 1\rangle.$$

Example. Consider S_7, $t7n7$ in the **Cayley** library. The procedure *schurmultiplier* gives us that this group has multiplier C_2 and the following presentation:

$$S_7 = \langle a,\ b \mid b^2 = a^7 = (a^2ba^{-2}b)^2 = (aba^{-1}b)^3 = (ab^{-1})^6 = 1\rangle.$$

From this we get the efficient presentation:

$$S_7 = \langle a,\ b \mid a^2b^7 = 1,\ (aba^{-1}b)^3 = (a^2ba^{-2}b)^2 = (a^{-1}b)^6\rangle.$$

Example. Using *schurmultiplier* we immediately see that the group:

$$t12n10 = \langle a,\ b,\ c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^2 = 1\rangle$$

is efficient since it has a presentation of deficiency three and its Schur multiplier is $C_2 \times C_2 \times C_2$. Similarly with the group:

$$\langle a,\ b \mid a^4 = (ab^{-1})^2 = b^4 = (ab)^6 = 1\rangle$$

since the presentation given by *schurmultiplier* for its multiplier defines $C_2 \times C_6$.

Section 2.3. Finding alternative presentations

If we are unsuccessful in obtaining an efficient presentation from the presentation given by *schurmultiplier* we can easily obtain other presentations to work with using **Cayley**. The following procedure will provide other presentations for groups that can be generated by two group elements:

```
procedure twogenpres(g);

r=relations(g);
cl=classes(g);
m=length(cl);
print '*****************************************'; print '';
for i=1 to m-1 do
  for j=i+1 to m do
   h=<cl[i],cl[j]>;
```

```
      if order(h) eq order(g) then
       rel=relations(h);
       car=card(rel);print car;
       print order(cl[i]),order(cl[j]),cl[i],cl[j];
       print rel; print '';
      end;
    end;
  end;
  print '****************************************';

  end;
```

The procedure *schurmultiplier* gives us that the group $t9n12$ has trivial multiplier and so an efficient presentation would have zero deficiency. However, the lowest deficiency presentation I could find was:

$$t9n12 = \langle a,\ b,\ c \mid a^2 = b^3 = c^3,\ abcacb = abac^{-1}bc = 1 \rangle.$$

In order to prove that a group is inefficient we must show that it is not possible to find an efficient presentation for the group. In Chapter 4 we show that the group $t9n12$ is inefficient.

Chapter 3. Some efficient direct products

Section 3.1. Efficiency of direct products

The efficiency of direct products of groups, stimulated by questions asked by Wiegold in [30], has been studied by several authors; see for example [1], [4], [7], [16]. In this chapter we give general methods for proving that direct products of two or three groups possessing certain properties are efficient and also give some specific examples. The most general of these examples involve the family of simple groups $PSL(2,\ p)$, for prime $p \geq 5$. $SL(2,\ p)$ is the group of two by two matrices having entries in $\mathcal{Z}_p$ of determinant one. This group has only one involution, the central element, and factoring by the centre yields $PSL(2,\ p)$. Both of these groups are perfect. $SL(2,\ p)$ has trivial Schur multiplier and $PSL(2,\ p)$ has multiplier C_2, its covering group being $SL(2,\ p)$.

The methods all hinge on a lemma of I. Miyamoto which is given below.

Lemma 3.1.1 (Lemma 2.1 of [5]). *Let G be a group with $a,\ b \in G$ satisfying $a^{\epsilon} = (a^m b^{\delta})^n$ where $\epsilon,\ \delta = \pm 1$, m and n integers. Then $\langle a,\ b\rangle$ is a cyclic subgroup of G and $a^{\epsilon - mn} = b^{\delta n}$.*

Proof. The relation $a^\epsilon = (a^m b^\delta)^n$ gives us that $[a,\ a^m b^\delta] = 1$ and so $[a,\ b] = 1$. Hence $\langle a,\ b\rangle$ is abelian. Also, $a = (a^m b^\delta)^{\epsilon n}$ and $b^\delta = (a^m b^\delta)^{1-\epsilon mn}$ which give $b = (a^m b^\delta)^{\delta(1-\epsilon mn)}$. Thus $\langle a,\ b\rangle$ is cyclic, generated by $a^m b^\delta$. The fact that $a^{\epsilon - mn} = b^{\delta n}$ comes from rewriting the relation using the fact that a and b commute. $\square$

This lemma can be extended to apply to three elements of a group as follows:

Lemma 3.1.2. (Lemma 3.1 of [5]). *Let G be a group and let $a,\ b,\ c \in G$ satisfy the relations*

$$a(ab^{-1})^2 = 1,\ c^\gamma = (c^k ab^{-1})^6$$

where $\gamma = \pm 1$, k an integer. Then $\langle a,\ b,\ c\rangle$ is cyclic and the relations $b^2 = a^3 = c^{6k-\gamma}$ hold in G.

Proof. The proof of Lemma 3.1.1 and the relation $a(ab^{-1})^2 = 1$ gives $a = (ab^{-1})^{-2}$, $b = (ab^{-1})^{-3}$. Furthermore, $\langle a,\ b\rangle$ is cyclic, generated by ab^{-1} as is $\langle c,\ ab^{-1}\rangle$ generated by $c^k ab^{-1}$. So we have $\langle a,\ b,\ c\rangle$ cyclic. Now $\langle a,\ b,\ c\rangle$ is abelian and $c^\gamma = (c^k ab^{-1})^6$ gives us that $c^{\gamma - 6k} = (ab^{-1})^6$ and the result follows. $\square$

In Section 3.2 we show that, for a certain set of groups, the direct product of any two is efficient. In Section 3.3 we show that the direct product of any three of them is also efficient. We use the following presentations:

$$PSL(2,\ p) = \langle a,\ b \mid a^2 = b^p = (ab)^3 = (ab^4ab^{(p+1)/2})^2 = 1\rangle$$

$$PSL(2,\ 25) = \langle a,\ b \mid a^2 = b^{13} = (ab)^3 = (ab^3ab^{-4})^2 = 1\rangle$$

$$PSL(2,\ 27) = \langle a,\ b \mid a^2 = b^{13} = (ab)^3 = (ab^3ab^{-3})^2 = 1\rangle$$

$$PSL(2,\ 49) = \langle a,\ b \mid a^2 = b^{25} = (ab)^3 = ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2} = 1\rangle$$

$$PSL(2,\ 81) = \langle a,\ b \mid a^2 = b^{41} = (ab)^3 = ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{-4} = 1\rangle$$

$$PSL(2,\ 169) = \langle a,\ b \mid a^2 = b^{85} = (ab)^3 = ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{-5} = 1\rangle.$$

The presentation for $PSL(2,\ p)$ comes from [24],those for $PSL(2,\ 25)$, $PSL(2,\ 27)$ were derived from those given in [6]. The presentations for $PSL(2,\ 49)$, $PSL(2,\ 81)$ and $PSL(2,\ 169)$ come from [5].

When dealing with the direct products involving $PSL(2,\ p)$ we invariably rely on the following result:

Lemma 3.1.3. *Let L be the group given by the presentation:*

$$\langle a,\ b \mid a^2 = s,\ (ab^4ab^{(p+1)/2})^2 = t,\ b^p = (ab)^3 = u;$$

$$s,\ t,\ u \text{ central involutions}\rangle,$$

for prime $p \geq 5$. Then $s = t$.

Proof. It is straightforward to check, using matrix methods described more fully in Section 1.7, that $L/L' \cong C_2$, b having order two in the abelianisation of L as follows. Let columns 1 and 2 represent a and b respectively in the following matrix:

$$\begin{pmatrix} 4 & 0 \\ 3 & 3-p \\ 0 & 2p \\ 8 & 2p+18 \end{pmatrix} \begin{matrix} row(4)-2row(1) \\ row(1)-row(2) \\ \longrightarrow \\ row(2)-3row(1) \end{matrix} \begin{pmatrix} 1 & p-3 \\ 0 & 12-4p \\ 0 & 2p \\ 0 & 2p+18 \end{pmatrix}.$$

Clearly $\text{hcf}(2p,\ 4p-12) = 2$. Therefore b has even order in L and u is non-trivial. Now $L/\langle u\rangle$ is a perfect group and is a stem extension of $PSL(2,\ p)$ with presentation:

$$\langle a,\ b \mid a^2 = s,\ (ab^4ab^{(p+1)/2})^2 = t,\ b^p = (ab)^3 = 1;$$
$$s,\ t \text{ central involutions}\rangle.$$

So, $L/\langle u\rangle$ is either $PSL(2,\ p)$ or $SL(2,\ p)$. In fact this group is $SL(2,\ p)$ since the following generators for $SL(2,\ p)$ satisfy these relations:

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We can check that $s = t$ in $L/\langle u\rangle$ using these matrix generators. Alternatively, we can consider the cases $s = 1$, $t \neq 1$ and $s \neq 1$, $t = 1$. In the case $s = 1$, since $SL(2,\ p)$ has only one involution, the central element, a is central and we have a contradiction since $SL(2,\ p)$ is not abelian. In the case $t = 1$ the central element would be $ab^4ab^{(p+1)/2}$, and so, factorising by this element, would give $PSL(2,\ p)$

with presentation:

$$\langle a,\ b \mid a^2 = 1,\ b^p = (ab)^3 = ab^4ab^{(p+1)/2} = 1\rangle.$$

Now we have $a^{-1}b^4a = b^{-(p+1)/2}$. Choose λ such that $4\lambda \equiv 1 \pmod{p}$. Then we have $a^{-1}ba = b^{-\lambda(p+1)/2}$ implying that $\langle b\rangle$ is a normal subgroup so again we have a contradiction.

Now we have $s = t$ in $L/\langle u\rangle$. Therefore $s \equiv t \pmod{u}$ in L. So, in L we have either $s = t$ or $t = su$. In the latter case, L would be given by the presentation:

$$\langle a,\ b \mid a^2 = s,\ (ab^4ab^{(p+1)/2})^2 = su,\ b^p = (ab)^3 = u;$$
$$s,\ u \text{ central involutions}\rangle.$$

Now if we factor out by s we obtain

$$\langle a,\ b \mid a^2 = 1,\ (ab^4ab^{(p+1)/2})^2 = b^p = (ab)^3 = u;$$
$$u \text{ a central involution}\rangle,$$

which is perfect. Since u is non-trivial this group cannot be $PSL(2,\ p)$. Using reasoning used earlier in the proof, this group cannot be $SL(2,\ p)$ either, since a would be the central involution forcing the group to be abelian. So we have a contradiction when $t = su$ and we must have $s = t$ in L as required. $\square$

Section 3.2. Efficient direct products of two groups

Theorem 3.2.1. *Let G_1, G_2 be finite perfect groups, both G_1 and G_2 having multiplier C_2. Let G_1, G_2 have presentations of the form*

$$\langle a,\ b \mid a^{\alpha_1} = b^{\alpha_2} = (ab)^{\alpha_3} = w(a,\ b) = 1\rangle$$

$$\langle x,\ y \mid x^{\beta_1} = y^{\beta_2} = (xy)^{\beta_3} = v(x,\ y) = 1\rangle$$

where the α_i, β_i satisfy the congruences $\beta_3 \equiv \pm 1 \pmod{\alpha_1}$, $\beta_2 \equiv \pm 1 \pmod{\alpha_3}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_3 \equiv \pm 1 \pmod{\beta_1}$. Let the group G, given by the presentation below, be perfect:

$$\langle a,\ b,\ x,\ y \mid a^{\alpha_1} = (xy)^{\pm 1}((xy)^{(\beta_3 \mp 1)/\alpha_1} a)^{\alpha_1} = y^{\pm 1}(y^{(\beta_2 \mp 1)/\alpha_3} ab)^{\alpha_3} =$$
$$(ab)^{\pm 1}((ab)^{(\alpha_3 \mp 1)/\beta_1} x)^{\beta_1} = b^{\pm 1}(b^{(\alpha_2 \mp 1)/\beta_1} x)^{\beta_1} = w(a,\ b)v(x,\ y) = 1\rangle.$$

Let G_1 be such that in the group presented by

$$\langle a,\ b \mid a^{\alpha_1} = 1,\ b^{\alpha_2} = (ab)^{\alpha_3} = u,\ w(a,\ b) = t;$$
$$u,\ t \text{ central involutions}\rangle$$

we have $t = 1$ and let G_2 be such that in the group presented by

$$\langle x,\ y \mid x^{\beta_1} = y^{\beta_2} = u,\ (xy)^{\beta_3} = v(x,\ y) = 1;\ u \text{ central involution}\rangle$$

we have $u = 1$. Then $G \cong G_1 \times G_2$ and so $G \cong G_1 \times G_2$ is efficient.

Proof. Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$. Lemma 3.1.1 gives us that $[xy,\ a] =$

$[y,\ ab] = [ab,\ x] = [b,\ x] = 1$ from which we get $[a,\ x] = [a,\ y] = [b,\ y] = 1$ and hence $[H,\ K] = 1$, and also that the following relations hold in G:

$$(xy)^{\beta_3} = 1,\ b^{\alpha_2} = (ab)^{\alpha_3} = x^{-\beta_1} = y^{-\beta_2}.$$

Let $D = \langle b^{\alpha_2},\ w(a,\ b)\rangle$ so that $D \leq H \cap K \leq Z(G)$. G is perfect and so by Lemma 1.5.1 we have that D is an epimorphic image of $M(G_1 \times G_2)$. Hence D is either trivial, C_2, or $C_2 \times C_2$. We have $G/D \cong G_1 \times G_2$, $H/D \cong G_1$ and $K/D \cong G_2$. Now, in H the following relations hold:

$$a^{\alpha_1} = 1,\ b^{\alpha_2} = (ab)^{\alpha_3} = u,\ w(a,\ b) = t,$$

with u, t, central involutions. By hypothesis we have $t = 1$ and so, by the sixth relation of G, $v(x,\ y) = 1$. Now K is given by

$$\langle x,\ y \mid x^{\beta_1} = y^{\beta_2} = u,\ (xy)^{\beta_3} = 1,\ v(x,\ y) = 1;\ u \text{ a central involution}\rangle$$

and so, by hypothesis we have $u = 1$. Hence D is trivial and $G \cong G_1 \times G_2$. $\square$

Corollary 3.2.2. *The direct product $G_1 \times G_2$ can be presented by*

$$\langle c,\ d \mid (c^{\pm\beta_2}d^{\pm\beta_1})^{\alpha_1} = (d^{\pm\alpha_2}c^{\mp\alpha_3})^{\pm 1}((d^{\pm\alpha_2}c^{\mp\alpha_3})^{(\beta_3\mp 1)/\alpha_1}c^{\pm\beta_2}d^{\pm\beta_1})^{\alpha_1}$$

$$= c^{\beta_2}(c^{\pm\beta_2(\alpha_3\mp 1)/\beta_1}d^{\pm\alpha_2})^{\beta_1} = w(c^{\pm\beta_2}d^{\pm\beta_1},\ d^{\mp\beta_1})v(d^{\pm\alpha_2},\ c^{\mp\alpha_3}) = 1\rangle$$

and so has an efficient presentation on two generators.

Proof. Put $c = y^{(\beta_2\mp 1)/\alpha_3}ab$, $d = b^{(\alpha_2\mp 1)/\beta_1}x$ so that $a = c^{\pm\beta_2}d^{\pm\beta_1}$, $b = d^{\mp\beta_1}$,

$x = d^{\pm\alpha_2}$, $y = c^{\mp\alpha_3}$ and substitute into the four generator presentation given in the theorem. $\Box$

We can easily modify this theorem to enable us to use these ideas when the groups G_1 and G_2 have slightly different presentations. Consider the conditions imposed on the α_i, β_j, i, $j \in \{1, 2, 3\}$ in Theorem 3.2.1. In any of the four congruences, we can interchange the α_i and β_j without affecting the main ideas of the proof. Also, let $\theta \in S_3$. Then, letting θ act on the i or j does not affect the mechanics of the proof. For example, suppose we have that $\beta_2 \equiv \pm 1 \pmod{\alpha_1}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_3 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_3}$ (this particular set of congruences was mentioned because they will hold in some of the examples given later). Such modifications are not only useful when the α_i, β_j do not satisfy the congruences of Theorem 3.2.1. We may wish to modify the theorem in order to make it possible to make G perfect or to obtain useful information from H and K which turns out to be the case in some of our examples. We can modify the theorem as follows:

Theorem 3.2.3. *Consider two groups, G_1 and G_2, having exactly the same properties as those required in Theorem 3.2.1 except that now the congruences the α_i, β_j, i, $j \in \{1, 2, 3\}$ satisfy are $\beta_2 \equiv \pm 1 \pmod{\alpha_1}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_3 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_3}$. Let G be defined as the group given by* *the presentation:*

$$\langle a,\ b,\ x,\ y \mid a^{\alpha_1} = y^{\pm 1}(y^{(\beta_2 \mp 1)/\alpha_1} a)^{\alpha_1} = b^{\pm 1}(b^{(\alpha_2 \mp 1)/\beta_1} x)^{\beta_1} =$$

$$(ab)^{\pm 1}((ab)^{(\alpha_3 \mp 1)/\beta_1} x)^{\beta_1} = b^{\pm 1}(b^{(\alpha_2 \mp 1)/\beta_3} xy)^{\beta_3} = w(a,\ b)v(x,\ y) = 1\rangle.$$

Let G be perfect and define H, K, and D as in Theorem 3.2.1. Also let H have the same property demanded of H in Theorem 3.2.1. Now if we have G_2 such that in the group presented by

$$\langle x,\ y \mid x^{\beta_1} = (xy)^{\beta_3} = u,\ y^{\beta_2} = v(x,\ y) = 1;$$

$$u \text{ a central involution}\rangle$$

we have $u = 1$ then G is the direct product of G_1 and G_2.

Corollary 3.2.4. *The direct product in Theorem 3.2.3 has the efficient presentation on two generators:*

$$\langle c,\ d \mid (c^{\mp\beta_1} d^{\pm\beta_3})^{\alpha_1} = d^{-\beta_3}(d^{\mp\beta_3(\alpha_2 \mp 1)/\beta_1} c^{\pm\alpha_3})^{\beta_1}$$

$$= (c^{\mp\alpha_3} d^{\pm\alpha_2})^{\pm 1}((c^{\mp\alpha_3} d^{\pm\alpha_2})^{(\beta_2 \mp 1)/\alpha_1} c^{\mp\beta_1} d^{\pm\beta_3})^{\alpha_1}$$

$$= w(c^{\mp\beta_1} d^{\pm\beta_3},\ d^{\mp\beta_3}) v(c^{\pm\alpha_3},\ c^{\mp\alpha_3} d^{\pm\alpha_2}) = 1\rangle.$$

Proof. Put $c = (ab)^{(\alpha_3 \mp 1)/\beta_1} x$, $d = b^{(\alpha_2 \mp 1)/\beta_3} xy$ and so $a = c^{\mp\beta_1} d^{\pm\beta_3}$, $b = d^{\mp\beta_3}$, $x = c^{\pm\alpha_3}$, $y = c^{\mp\alpha_3} d^{\pm\alpha_2}$ and substitute into the presentation in Theorem 3.2.3. □

Example 3.2.5. *$PSL(2,\ p) \times PSL(2,\ 25)$, p prime ≥ 5, is efficient with presentation:*

$$\langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = y(y^4ab)^3 = ab(abx)^2 =$$
$$b(b^{(p-1)/2}x)^2 = (ab^4ab^{(p+1)/2})^2(xy^3xy^{-4})^2 = 1\rangle.$$

Proof. Let G be the group defined by this presentation and $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$. By Lemma 3.1.1 we have $[H,\ K] = 1$ and the following relations in G:

$$(xy)^3 = 1,\ b^p = (ab)^3 = x^{-2} = y^{-13}.$$

Let $D = \langle b^p,\ w(a,\ b)\rangle$ so we have $D \leq H \cap K \leq Z(G)$. We can show that G is perfect using matrix methods as described in Section 1.7. Consider the abelianisation of G as a matrix with each column corresponding to a generator and each row corresponding to a relator, the element in position $(i,\ j)$ corresponding to the index of generator j in relator i. Then Tietze transformations are equivalent to row operations on the matrix and we can show that this abelianisation is trivial, and hence the group is perfect, by applying such operations until the matrix has exactly one 1 in each column. This corresponds to each generator having order one in the abelianisation. In this case, calculations are made simplest by having columns one to four representing $x,\ y,\ a,\ b$ respectively. We begin with the matrix:

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 3 & 3 & 2 & 0 \\ 0 & 13 & 3 & 3 \\ 2 & 0 & 3 & 3 \\ 2 & 0 & 0 & p \\ 4 & -2 & 4 & p+9 \end{pmatrix}$$

and proceed as follows:

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 3 & 3 & 2 & 0 \\ 0 & 13 & 3 & 3 \\ 2 & 0 & 3 & 3 \\ 2 & 0 & 0 & p \\ 4 & -2 & 4 & p+9 \end{pmatrix} \begin{array}{c} row(2)-row(4) \\ row(5)-row(4) \\ \longrightarrow \\ row(6)-row(4) \\ row(4)-2row(2) \end{array} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 3 & -1 & -3 \\ 0 & 13 & 3 & 3 \\ 0 & -6 & 5 & 9 \\ 0 & 0 & -3 & p-3 \\ 0 & -2 & -2 & p+3 \end{pmatrix}.$$

Now we can omit the first column and second row since these no longer affect the calculations and continue as follows:

$$\begin{pmatrix} 0 & 2 & 0 \\ 13 & 3 & 3 \\ -6 & 5 & 9 \\ 0 & -3 & p-3 \\ -2 & -2 & p+3 \end{pmatrix} \begin{array}{c} row(2)+2row(3) \\ row(3)+6row(2) \\ \longrightarrow \\ row(5)+2row(2) \end{array} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 13 & 21 \\ 0 & 83 & 135 \\ 0 & -3 & p-3 \\ 0 & 24 & p+45 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 83 & 135 \\ -3 & p-3 \\ 24 & p+45 \end{pmatrix} \begin{array}{c} row(2) - 41row(1) \\ row(4) - 12row(1) \\ 2row(1) + row(3) \\ \longrightarrow \\ row(3) + 3row(1) \\ row(2) - row(1) \end{array} \begin{pmatrix} 1 & p-3 \\ 0 & 138-p \\ 0 & 4p-12 \\ 0 & p+45 \end{pmatrix}.$$

This abelianisation is trivial if we can get 1 as a linear combination of $p-138$, $4p-12$, $p+45$. We can get $183 = 3.61$ by taking the difference of the first and third of these. We can get $192 = 2^6.3$ by combining the second and third. So, we can get 3 as a linear combination of these three numbers. This means that we can also get p. Now, 3 and p are coprime and so we can indeed get 1 as a linear combination of these three numbers and our original group is perfect. Now, by Lemma 1.5.1, D must be an epimorphic image of $M(PSL(2,\ p) \times PSL(2,\ 25))$. Hence D is either trivial, C_2 or $C_2 \times C_2$. Now in H the following relations hold:

$$a^2 = 1,\ b^p = (ab)^3 = u,\ (ab^4ab^{(p+1)/2})^2 = t,$$

with u, t central involutions. By Lemma 3.1.3 we have $t = 1$ and so, by the sixth relation of G, $(xy^3xy^{-4})^2 = 1$. Now K is given by:

$$\langle x,\ y \mid x^2 = y^{13} = u,\ (xy)^3 = (xy^3xy^{-4})^2 = 1,\ u \text{ a central involution}\rangle.$$

We can easily check using **Cayley** or **GAP** that $u = 1$ in K. Hence D is trivial and $G \cong PSL(2,\ p) \times PSL(2,\ 25)$ as required. $\square$

We can also write down an efficient presentation for $PSL(2,\ p) \times PSL(2,\ 25)$ on two generators using Corollary 3.2.2:

$$\langle c,\ d \mid (c^{13}d^2)^2 = d^p c^{-3}(d^p c^{10} d^2)^2 = c^{13}(c^{13}d^p)^2$$
$$= (c^{13}d^{-6}c^{13}d^{-(p+3)})^2(d^p c^{-9} d^p c^{12})^2 = 1\rangle.$$

Example 3.2.6. *$PSL(2,\ 81)^2$ is efficient with presentation:*

$$\langle a,\ b,\ x,\ y \mid a^2 = y(y^{20}a)^2 = b(b^{20}x)^2 = ab(abx)^2 = b^{-1}(b^{14}xy)^3$$
$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}x^{-1}y^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4} = 1\rangle.$$

Proof. The proof works in accordance with Theorem 3.2.3 in the same way that Example 3.2.5 does with Theorem 3.2.1. □

Using Corollary 3.2.4 we can also obtain an efficient presentation for $PSL(2,\ 81)^2$ on two generators:

$$\langle c,\ d \mid (c^{-2}d^{-3})^2 = c^{-3}d^{-41}((c^{-3}d^{-41})^{20}c^{-2}d^{-3})^2 = d^3(d^{60}c^3)^2$$
$$= c^{-2}d^3c^{-2}d^{-15}c^{-4}d^{-12}c^{-2}d^{12}c^{-2}d^{-12}c^{-4}d^{108}.$$
$$c^{-3}(c^{-3}d^{-41})^2c^3(d^{41}c^3)^4d^{-41}c^3(d^{41}c^3)^3c^3(c^{-3}d^{-41})^5c^3(d^{41}c^3)^3d^{-41}c^3(d^{41}c^3)^4 = 1\rangle.$$

Theorem 3.2.7. *$PSL(2,\ p)^2$, p prime ≥ 5, is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y^{\pm 1}(y^{(p\mp 1)/3}ab)^3$$
$$= (ab^{-4}ab^{(p-1)/2})^2(xy^{-4}xy^{(p-1)/2})^2 = 1\rangle.$$

The method is a variation on the proof that $PSL(2,\ p)^2$ is efficient in [4].

Proof. We use the presentation:

$$PSL(2,\ p) = \langle a,\ b \mid a^2 = b^p = (ab)^3 = (ab^{-4}ab^{(p-1)/2})^2 = 1\rangle.$$

Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$, $D = \langle b^p,\ (ab^{-4}ab^{(p-1)/2})^2\rangle$. D is central and we can use the ideas of Section 1.7 to show that G is perfect in this case. Letting columns one to four represent generators $a,\ b,\ x,\ y$ respectively in a matrix representing the abelianisation of G we proceed as follows:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & 3 & 3 \\ 0 & p & 2 & 0 \\ 3 & 3 & 2 & 0 \\ 3 & 3 & 0 & p \\ 4 & p-9 & 4 & p-9 \end{pmatrix} \quad \begin{matrix} row(2) - row(1) \\ row(6) - 2row(1) \\ row(5) - row(4) \\ \longrightarrow \\ row(4) - row(1) \\ row(1) - 2row(4) \end{matrix} \quad \begin{pmatrix} 0 & -6 & -4 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & p & 2 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 0 & -2 & p \\ 0 & p-9 & 4 & p-9 \end{pmatrix},$$

$$\begin{pmatrix} -6 & -4 & 0 \\ 0 & 3 & 3 \\ p & 2 & 0 \\ 0 & -2 & p \\ p-9 & 4 & p-9 \end{pmatrix} \quad \begin{matrix} row(5) - row(3) \\ row(5) - row(1) \\ \longrightarrow \\ row(1) - 2row(5) \end{matrix} \quad \begin{pmatrix} 0 & -16 & 18-p \\ 0 & 3 & 3 \\ p^* & 2 & 0 \\ 0 & -2 & p \\ -3 & 6 & p-9 \end{pmatrix}.$$

Now we wish to put a 1 in position (*) and so must consider two separate cases, the case when $p = 3m + 1$, for some integer m, and the case when $p = 3m - 1$. Consider firstly the case when $p = 3m + 1$:

$$\begin{pmatrix} 0 & -16 & 18-p \\ 0 & 3 & 3 \\ p & 2 & 0 \\ 0 & -2 & p \\ -3 & 6 & p-9 \end{pmatrix} \begin{matrix} row(3) \\ +mrow(5) \\ \longrightarrow \\ row(5) \\ +3row(3) \end{matrix} \begin{pmatrix} 0 & -16 & 18-p \\ 0 & 3 & 3 \\ 1 & 2+6m & m(p-9) \\ 0 & -2 & p \\ 0 & 12+18m & (p-9)(1+3m) \end{pmatrix},$$

$$\begin{pmatrix} -16 & 18-p \\ 3 & 3 \\ -2 & p \\ 12+18m & (p-9)(1+3m) \end{pmatrix} \begin{matrix} row(1)-8row(3) \\ row(4)+ \\ (6+9m)row(3) \\ \longrightarrow \\ row(2)+row(3) \\ row(3)+2row(2) \end{matrix} \begin{pmatrix} 0 & 18-9p \\ 1 & 3+p \\ 0 & 3p+6 \\ 0 & 4p^2-6p \end{pmatrix}.$$

Now we consider the case $p = 3m - 1$:

$$\begin{pmatrix} 0 & -16 & 18-p \\ 0 & 3 & 3 \\ p & 2 & 0 \\ 0 & -2 & p \\ -3 & 6 & p-9 \end{pmatrix} \begin{matrix} row(3)+ \\ mrow(5) \\ \longrightarrow \\ row(5)+ \\ 3row(3) \end{matrix} \begin{pmatrix} 0 & -16 & 18-p \\ 0 & 3 & 3 \\ -1 & 2+6m & m(p-9) \\ 0 & -2 & p \\ 0 & -18m & (p-9)(1-3m) \end{pmatrix},$$

$$\begin{pmatrix} -16 & 18-p \\ 3 & 3 \\ -2 & p \\ -18m & (p-9)(1-3m) \end{pmatrix} \begin{array}{c} row(1)-8row(3) \\ row(4)-(9m)row(3) \\ \longrightarrow \\ row(2)+row(3) \\ row(3)+2row(2) \end{array} \begin{pmatrix} 0 & 18-9p \\ 1 & 3+p \\ 0 & 3p+6 \\ 0 & 6p-4p^2 \end{pmatrix}.$$

So both cases reduce to the problem of writing 1 as a linear combination of $3p+6$, $9p-18$, $4p^2-6p$. We can easily get 36 as a linear combination of the first two. Also both of these first two are odd and so we can get 9 as a linear combination of the first two. Now $4p^2-6p$ is coprime to 9 and so we have our result and the group G is perfect. So, from Lemma 1.5.1, we can deduce that D is an epimorphic image of $C_2 \times C_2$. Also we know that $G/D \cong PSL(2,\ p)^2$ and $H/D \cong K/D \cong PSL(2,\ p)$. The following relations hold in H:

$$a^2 = 1,\ b^p = (ab)^3 = u,\ (ab^{-4}ab^{(p-1)/2})^2 = t;$$

$$u,\ t \text{ central involutions.}$$

We want to show that these relations imply $t=1$. Assume otherwise and let H_1 be the group defined by these relations and consider its abelianisation in matrix form, columns one, two representing a, b respectively.

$$\begin{pmatrix} 2 & 0 \\ 3 & 3-p \\ 0 & 2p \\ 8 & 2p-18 \end{pmatrix} \begin{array}{c} row(4)-4row(1) \\ row(2)-row(1) \\ \longrightarrow \\ row(1)-2row(2) \end{array} \begin{pmatrix} 0 & 2p-6 \\ 1 & 3-p \\ 0 & 2p \\ 0 & 2p-18 \end{pmatrix}.$$

Since all entries in column two of this matrix are even and $2p$ and 18 have highest common factor 2 then in the abelianisation of this group, b has order two so u is not trivial. The group obtained by factoring H_1 by $\langle u \rangle$ is perfect and so must be $SL(2,\ p)$ or $PSL(2,\ p)$. We get a contradiction if we assume the group is $SL(2,\ p)$ since a would be the central involution, forcing the group to be abelian. So t must be trivial in $H_1/\langle u \rangle$. Now in H_1 we must have either $t = 1$ or $t = u$. Assume that $t = u$. We then have H_1 perfect and, since u is non trivial, we again obtain a contradiction since H_1 cannot be $SL(2,\ p)$. Hence in H_1 we must have $t = 1$, $u \neq 1$ and t is trivial in H. Now K is given by:

$$K = \langle x,\ y \mid x^2 = y^p = u,\ (xy)^3 = (xy^{-4}xy^{(p-1)/2})^2 = 1;$$
$$u \text{ a central involution}\rangle.$$

We can easily check that this group is perfect by constructing a matrix with columns one and two corresponding to x and y respectively:

$$\begin{pmatrix} 2 & -p \\ 4 & 0 \\ 3 & 3 \\ 4 & p-9 \end{pmatrix} \xrightarrow[\substack{row(3)-3row(2) \\ row(4)-4row(2)}]{\substack{row(2)-row(3) \\ row(1)-2row(2)}} \begin{pmatrix} 0 & -p+6 \\ 1 & 3 \\ 0 & 12 \\ 0 & p+3 \end{pmatrix}$$

(we can get a 1 in the column corresponding to y since 12 and $p+3$ are coprime). So K must be either $SL(2,\ p)$ or $PSL(2,\ p)$. If it were $SL(2,\ p)$ then $PSL(2,\ p)$

would be presented by:

$$F = \langle x,\ y \mid x^2 = y^p = (xy)^3 = xy^{-4}xy^{(p-1)/2} = 1\rangle.$$

So, in F, $x^{-1}y^4x = y^{(p-1)/2}$. We can choose λ such that $4\lambda \equiv 1 \pmod p$ since $(p,\ 4) = 1$. Then we have $x^{-1}yx = y^{\lambda(p-1)/2}$ implying that $\langle y\rangle$ is a normal subgroup and we have a contradiction. So K must be $PSL(2,\ p)$, D trivial, and $G \cong PSL(2,\ p)^2$. Hence $PSL(2,\ p)^2$ is efficient $\square$

We can write down an efficient presentation for $PSL(2,\ p)^2$ on two generators using Corollary 3.2.2 as follows:

$$\begin{aligned}\langle c,\ d \mid (c^{\pm p}d^2)^2 &= d^p c^{\mp 3}(d^p c^{\pm(p-3)}d^2)^2 = c^{\pm p}(c^{\pm p}d^p)^2 \\ &= (c^{\pm p}d^{10}c^{\pm p}d^{-p+3})^2(d^p c^{\pm 12}d^p c^{\mp 3(p-1)/2})^2 = 1\rangle.\end{aligned}$$

Theorem 3.2.8. *The direct product of any two of $PSL(2,\ p)$, p prime ≥ 5, $PSL(2,\ 25)$, $PSL(2,\ 27)$, $PSL(2,\ 49)$, $PSL(2,\ 81)$, $PSL(2,\ 169)$ is efficient.*

Proof. We can use Theorem 3.2.1 and Theorem 3.2.3 to construct four generator presentations for these direct products. The occurrance of (*) after any of the following presentations indicates that Theorem 3.2.3 leads to that presentation.

$PSL(2,\ p) \times PSL(2,\ p)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y^{\pm 1}(y^{(p\mp 1)/3}ab)^3$$

$$= (ab^{-4}ab^{(p-1)/2})^2(xy^{-4}xy^{(p-1)/2})^2 = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 25)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= (ab^4ab^{(p+1)/2})^2(xy^3xy^{-4})^2 = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= (ab^4ab^{(p+1)/2})^2(xy^3xy^{-3})^2 = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y(y^8ab)^3$$

$$= (ab^4ab^{(p+1)/2})^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y^{-1}(y^{14}ab)^3$$

$$= (ab^4ab^{(p+1)/2})^2x^{-1}y^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y(y^{28}ab)^3$$

$$= (ab^4ab^{(p+1)/2})^2x^{-1}y^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 25)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^4ab)^3$$
$$= (ab^3ab^{-4})^2(xy^3xy^{-4})^2 = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^4ab)^3$$
$$= (ab^3ab^{-4})^2(xy^3xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^8ab)^3$$
$$= (ab^3ab^{-4})^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y^{-1}(y^{14}ab)^3$$
$$= (ab^3ab^{-4})^2x^{-1}y^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^{28}ab)^3$$
$$= (ab^3ab^{-4})^2x^{-1}y^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^4ab)^3$$
$$= (ab^3ab^{-3})^2(xy^3xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^8ab)^3$$
$$= (ab^3ab^{-3})^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y^{-1}(y^{14}ab)^3$$
$$= (ab^3ab^{-3})^2x^{-1}y^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^{28}ab)^3$$
$$= (ab^3ab^{-3})^2x^{-1}y^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{12}x)^2 = ab(abx)^2 = y(y^8ab)^3$$
$$= ab^2ab^{23}ab^4ab^{18}ab^4ab^{23}x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{20}x)^2 = ab(abx)^2 = y^{-1}(y^{14}ab)^3$$
$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{42}x)^2 = ab(abx)^2 = y(y^8ab)^3$$
$$ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1\rangle$$

$PSL(2,\ 81) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = y(y^{20}a)^2 = ab(abx)^2 = b(b^{20}x)^2 = b^{-1}(b^{14}xy)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}x^{-1}y^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4} = 1\rangle \quad (*)$$

$PSL(2,\ 81) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = xy(xya)^2 = b(b^{20}x)^2 = ab(abx)^2 = y(y^{14}ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}x^{-1}y^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5} = 1\rangle$$

$PSL(2,\ 169) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y \mid a^2 = y(y^{42}a)^2 = ab(abx)^2 = b(b^{42}x)^2 = b(b^{28}xy)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}x^{-1}y^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5} = 1\rangle \quad (*)$$

In many of these direct product presentations, $w(a,\ b)$ and/or $v(x,\ y)$ are modified. This is to accommodate the $t = 1,\ u = 1$ properties required of $G_1,\ G_2$ respectively in Theorems 3.2.1 and 3.2.3. and also to make sure that G is perfect. In each case, these are the only three things we need to check to verify that the direct product is efficient. □

Section 3.3. Efficient direct products of three groups.

Theorem 3.3.1. *Let $G_1,\ G_2,\ G_3$ be finite perfect groups, each with multiplier*

C_2. *Let* G_1, G_2, G_3 *have presentations*

$$\langle a,\ b \mid a^2 = b^l = (ab)^3 = w_1(a,\ b) = 1\rangle$$

$$\langle x,\ y \mid x^2 = y^m = (xy)^3 = w_2(x,\ y) = 1\rangle$$

$$\langle u,\ v \mid u^2 = v^n = (uv)^3 = w_3(u,\ v) = 1\rangle$$

respectively where $l,\ m,\ n \equiv \pm 1 \pmod 6$. *Let* G_1, G_2, G_3 *be such that in each of the group presentations below we have* $s = t$*:*

$$\langle a,\ b \mid a^2 = b^l = (ab)^3 = s,\ w_1(a,\ b) = t;\ s,\ t \text{ central involutions}\rangle$$

$$\langle x,\ y \mid x^2 = y^m = (xy)^3 = s,\ w_2(x,\ y) = t;\ s,\ t \text{ central involutions}\rangle$$

$$\langle u,\ v \mid u^2 = v^n = (uv)^3 = s,\ w_3(u,\ v) = t;\ s,\ t \text{ central involutions}\rangle.$$

Let G *be the group given by*

$$\langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(n\mp 1)/6}xya^{-1})^6 =$$
$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(l\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(m\mp 1)/6}abu^{-1})^6 =$$
$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(l\mp 1)/6}xyu^{-1})^6 = w_1(a,\ b)^{-1}w_2(x,\ y)w_3(u,\ v) = 1\rangle.$$

Then, if G *is perfect, we have* $G \cong G_1 \times G_2 \times G_3$ *and so this direct product is efficient.*

Proof. The proof is similar to the proof in [5] where it is proved that $PSL(2,\ p)^3$, $p \geq 5$ is efficient. Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$, $L = \langle u,\ v\rangle$ and $D = \langle a^2,\ w_1(a,\ b),\ w_2(x,\ y)\rangle$. By Lemma 3.1.2 we have $[H,\ K] = [H,\ L] = [K,\ L] =$

1 and the following relations holding in G:

$$v^n = a^2 = (xy)^3 = b^l = x^2 = (uv)^3 = y^m = u^2 = (ab)^3.$$

These relations, along with the ninth relation of G, give us that $D \leq Z(G)$ since all the generators of D can also be written in terms of other pairs of generators of G. G is perfect and so, by Lemma 1.5.1, D is a homomorphic image of $C_2 \times C_2 \times C_2$. Since we have $s = t$ in

$$\langle a,\ b \mid a^2 = b^l = (ab)^3 = s,\ w_1(a,\ b) = t;\ s,\ t \text{ central involutions}\rangle$$

we have $a^2 = w_1(a,\ b)$ in G and similarly $x^2 = w_2(x,\ y)$, $u^2 = w_3(u,\ v)$. We also have $a^2 = x^2 = u^2$ and so, by the ninth relation of G, we have $a^2 = w_1(a,\ b) = x^2 = w_2(x,\ y) = u^2 = w_3(u,\ v) = 1$. Hence $D = 1$ and $G \cong G_1 \times G_2 \times G_3$ as required. □

Corollary 3.3.2. *The direct product in Theorem 3.3.1 also has an efficient presentation on two generators.*

Proof. We work from the six generator presentation given in Theorem 3.3.1. Let $c = v^{(n\mp1)/6}xya^{-1}$. The first two relations give us that $v = c^{\mp6}$, $xya^{-1} = c^{\pm n}$, $xy = c^{\mp2n}$, $a = c^{\mp3n}$. Now let $d = b^{(l\mp1)/6}uvx^{-1}$. Then the third and fourth relations give us that $b = c^{\mp6}$, $uvx^{-1} = d^{\pm l}$, $uv = d^{\mp2l}$, $x = d^{\mp3l}$. So, all we need now is that $y = d^{\pm3l}c^{\mp2n}$ and $u = d^{\mp2l}c^{\pm6}$ and we can rewrite the six generator presentation in terms of c and d alone, the first four relations becoming trivial.

So, an efficient two generator presentation for $G_1 \times G_2 \times G_3$ is:

$$\langle c,\ d \mid c^{\mp 3n}d^{\mp 6}(c^{\mp 3n}d^{\mp 6}c^{\mp 6}d^{\pm 2l})^2 = (d^{\pm 3l}c^{\mp 2n})^{\pm 1}((d^{\pm 3l}c^{\mp 2n})^{(m\mp 1)/6}c^{\mp 3n}d^{\mp 6}.$$

$$c^{\mp 6}d^{\pm 2l})^6 = c^{\mp 2n}(c^{\mp 2n \mp 6}d^{\pm 2l})^2 = d^{-6}(d^{\mp l+1}c^{\mp 2n\mp 6}d^{\pm 2l})^6$$

$$= w_1(c^{\mp 3n},\ c^{\mp 6})^{-1}w_2(d^{\mp 3l},\ d^{\pm 3l}c^{\mp 2n})w_3(d^{\mp 2l}c^{\pm 6},\ c^{\mp 6}) = 1\rangle.$$

□

We can also modify the theorem to find efficient presentations for direct products of three groups including one or more with a presentation of the form

$$\langle a,\ b \mid a^2 = b^l = (ab)^3 = w(a,\ b) = 1\rangle$$

where $l \equiv \pm 1 \pmod 6$ and rather than have $s = t$ in

$$\langle a,\ b \mid a^2 = b^l = (ab)^3 = s,\ w(a,\ b) = t;\ s,\ t \text{ central involutions}\rangle$$

we have $s = 1$ or $t = 1$. This is easily done by modifying the ninth relation of G.

Example 3.3.3. *$PSL(2,\ 25) \times PSL(2\ 27) \times PSL(2,\ p)$, p prime ≥ 5, is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{(}b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^3ab^{-4})^2(xy^3xy^{-3})^2(uv^4uv^{(p+1)/2}uv^4u^{-17}v^{(p+1)/2})^{-1} = 1\rangle.$$

Proof. Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$, $L = \langle u,\ v\rangle$. The first eight relations give us, from Lemma 3.1.2, that $[H,\ K] = [H,\ L] = [L,\ K] = 1$ and that

$$v^p = a^2 = (xy)^3 = b^{13} = x^2 = (uv)^3 = y^{13} = u^2 = (ab)^3.$$

Let $D = \langle a^2,\ (ab^3ab^{-4})^2,\ (xy^3xy^{-3})^2\rangle$. Now $G/D \cong PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ p)$ which has trivial centre so $D = Z(G)$. The ninth relation of G may not appear to follow naturally from Theorem 3.3.1 and the presentation for $PSL(2,\ p)$ given in Section 3.1. This is because the presentation that would follow most naturally does not define a perfect group. However, since we wish u to have order two in G, we can add multiples of two to the index of a anywhere in this relation in the hope of making G perfect (we also have similar degrees of freedom corresponding to the other generators but it does not prove necessary to take advantage of these). Let $-q$ be the index of u in this ninth relation and consider the matrix representing the abelianisation of G, columns one to six representing generators a, b, x, y, u, v respectively:

$$\begin{pmatrix} -2 & 13 & 0 & 0 & 0 & 0 \\ 0 & 13 & -2 & 13 & 0 & 0 \\ 0 & 0 & -2 & 13 & 0 & 0 \\ 0 & 0 & 0 & 13 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & p \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 4 & -2 & 4 & 0 & -q & -p-9 \end{pmatrix} \quad \begin{matrix} row(9) + 2row(1) \\ \longrightarrow \\ row(1) + 2row(6) \end{matrix}$$

$$\begin{pmatrix} 0 & 19 & 0 & 0 & 0 & 0 \\ 0 & 13 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 13 & 0 & 0 \\ 0 & 0 & 0 & 13 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & p \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 24 & 4 & 0 & -q & -p-9 \end{pmatrix},$$

$$\begin{pmatrix} 19 & 0 & 0 & 0 & 0 \\ 13 & -2 & 0 & 0 & 0 \\ 0 & -2 & 13 & 0 & 0 \\ 0 & 0 & 13 & -2 & 0 \\ 0 & 0 & 0 & -2 & p \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 24 & 4 & 0 & -q & -p-9 \end{pmatrix} \begin{matrix} 3row(2) \\ -2row(1) \\ row(8) \\ -row(1) \\ \longrightarrow \\ row(8) \\ -6row(2) \end{matrix} \begin{pmatrix} 19 & 0 & 0 & 0 & 0 \\ 1 & -6 & 0 & 0 & 0 \\ 0 & -2 & 13 & 0 & 0 \\ 0 & 0 & 13 & -2 & 0 \\ 0 & 0 & 0 & -2 & p \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 34 & 0 & -q & -p-9 \end{pmatrix}.$$

Now $row(1)$ is redundant and so we can omit it from further calculations.

$$\begin{pmatrix} -2 & 13 & 0 & 0 \\ 0 & 13 & -2 & 0 \\ 0 & 0 & -2 & p \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 34 & 0 & -q & -p-9 \end{pmatrix} \begin{matrix} row(1) + 2row(6) \\ \longrightarrow \\ row(6) - 34row(4) \end{matrix} \begin{pmatrix} 0 & 19 & 0 & 0 \\ 0 & 13 & -2 & 0 \\ 0 & 0 & -2 & p \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -102 & -q & -p-9 \end{pmatrix},$$

$$\begin{pmatrix} 19 & 0 & 0 \\ 13 & -2 & 0 \\ 0 & -2 & p \\ 0 & 1 & 3 \\ -102 & -q & -p-9 \end{pmatrix} \begin{matrix} -row(5) \\ 3row(2) - 2row(1) \\ \longrightarrow \\ row(5) - 5row(1) \\ row(5) - 7row(2) \end{matrix} \begin{pmatrix} 19 & 0 & 0 \\ 1 & -6 & 0 \\ 0 & -2 & p \\ 0 & 1 & 3 \\ 0 & q+42 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} -2 & p \\ 1 & 3 \\ q+42 & p+9 \end{pmatrix} \quad \begin{matrix} row(1)+2row(2) \\ \longrightarrow \\ row(3)-(q+42)row(2) \end{matrix} \quad \begin{pmatrix} 0 & p+6 \\ 1 & 3 \\ 0 & p-3q-117 \end{pmatrix}.$$

Now we wish to get 1 as a linear combination of $p+6$ and $p-3q-117$. We can easily get $3q+123$. Now $p+6$ obviously has no factor 3, so if we choose q such that $3q+123$ is a power of 3 then we are done and G is perfect ($p+6$ is also obviously coprime to 2 but we cannot make $3q+123$ a power of 2 since q must be even). Putting $q=-14$ makes $3q+123=81$. Hence G is perfect. Now using Lemma 1.5.1 we have D is a homomorphic image of $M(PSL(2,\ 25)\times PSL(2,\ 27)\times PSL(2,\ p))\cong C_2\times C_2\times C_2$. Now

$$H=\langle a,\ b\mid a^2=b^{13}=(ab)^3=s,\ (ab^3ab^{-4})^2=t_1;$$

$$s,\ t_1 \text{ central involutions}\rangle$$

$$K=\langle x,\ y\mid x^2=y^{13}=(xy)^3=s,\ (xy^3xy^{-3})^2=t_2;$$

$$s,\ t_2 \text{ central involutions}\rangle$$

$$L=\langle u,\ v\mid u^2=v^p=(uv)^3=s,\ uv^4uv^{(p+1)/2}uv^4u^{-17}v^{(p+1)/2}=t_3;$$

$$s,\ t_3 \text{ central involutions}\rangle.$$

We can easily check by computer that $s=t_i$ for $i=1,2$ using **Cayley** or **GAP** and it follows from Lemma 3.1.3 that this is also the case for $i=3$. So $(ab^3ab^{-4})^2=(xy^3xy^{-3})^2=uv^4uv^{(p+1)/2}uv^4u^{-17}v^{(p+1)/2}=1$ by the ninth relation of G. Now $s=1$ and so D is trivial. Hence we have that $G\cong PSL(2,\ 25)\times PSL(2,\ 27)\times PSL(2,\ p)$ and so $PSL(2,\ 25)\times PSL(2,\ 27)\times PSL(2,\ p)$ is efficient as required. $\square$

We can get an efficient presentation for $PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ p)$ on two generators by putting $c = b^2uvx^{-1}$, $d = y^2abu^{-1}$. The third, fourth, fifth and sixth relations then give us that $a = d^{-26}c^6$, $b = c^{-6}$, $x = c^{-39}$, $y = d^{-6}$, $u = d^{-39}$, $v = d^{39}c^{-26}$:

$$PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ p) =$$

$$\langle c,\ d \mid c^{-39}d^{-6}(c^{-39}d^{-6}c^{-6}d^{26})^2 = (d^{39}c^{-26})^{\pm 1}((d^{39}c^{-26})^{(p\mp 1)/6}c^{-39}d^{-6}c^{-6}d^{26})^6$$

$$= c^{-39}d^{-6}(c^{-39}d^{33})^2 = c^{-6}(c^{-51}d^{33})^6$$

$$= (d^{-26}c^{-12}d^{-26}c^{30})^2(c^{-39}d^{18}c^{-39}d^{18})^2(d^{-39}(d^{39}c^{-26})^4.$$

$$d^{-39}(d^{-39}(d^{39}c^{-26})^{(p+1)/2}d^{-39}(d^{39}c^{-26})^4d^{663}(d^{39}c^{-26})^{(p+1)/2})^{-1} = 1\rangle.$$

Example 3.3.4. *$PSL(2,\ 49)^3$ is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 =$$

$$y(y^4abu^{-1})^6 = xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}uv^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2} = 1\rangle.$$

Proof. Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$, $L = \langle u,\ v\rangle$. By Lemma 3.1.2 we have

$[H,\ K] = [H,\ L] = [K,\ L] = 1$ and the following relations in G:

$$v^{25} = a^2 = (xy)^3 = b^{25} = x^2 = (uv)^3 = y^{25} = u^2 = (ab)^3.$$

Let $D = \langle a^2,\ a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2},\ xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}\rangle$. Then $G/D \cong PSL(2,\ 49)^3$ and $D \leq Z(G)$. We can easily check using matrix methods, **Cayley** or **GAP** that G is perfect and so D must be an epimorphic image of $C_2 \times C_2 \times C_2$. Now, in H, the following relations hold:

$$a^2 = b^{25} = (ab)^3 = s,\ ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2} = t;$$

$$s,\ t \text{ central involutions},$$

and similarly in K and L. Let H_1 be the group defined by these relations. We want to show that t is trivial in H_1. H_1 is perfect so must be either $SL(2,\ 49)$ or $PSL(2,\ 49)$. Now we must have either $s = 1,\ t \neq 1$ or $s \neq 1,\ t = 1$ or $s = t$. Coset enumeration of H_1 over $\langle b^2\rangle$ gives us index 4704. Addition of the relation $t = 1$ does not affect the index and so we must have $t = 1$ as required. Hence $ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2} = 1$ in G. Similarly $xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} = 1$ and $uv^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2} = 1$ in G. So, in G, we have

$$\begin{aligned} ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2} &= xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2} \\ &= uv^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2} = 1. \end{aligned}$$

Using these along with the ninth relation in the presentation of G we obtain $a^2 = 1$. Hence $D = 1$ and $G \cong PSL(2,\ 49)^3$ as required. $\square$

We can easily obtain an efficient two generator presentation for $PSL(2,\ 49)^3$ using Corollary 3.3.2. Letting $c = v^4xya^{-1}$, $d = b^4uvx^{-1}$ we obtain the following presentation:

$$\langle c,\ d \mid c^{-75}d^{-6}(c^{-75}d^{-6}c^{-6}d^{50})^2 = d^{75}c^{-50}((d^{75}c^{-50})^4c^{-75}d^{-6}c^{-6}d^{50})^6$$

$$= c^{-50}(c^{-56}d^{50})^2 = d^{-6}(d^{-24}c^{-56}d^{50})^6$$

$$= c^{93}d^{-12}c^{-75}d^{12}c^{-75}d^{-24}c^{-75}d^{42}c^{-75}d^{-24}c^{-75}d^{12}c^{-50}d^{75}c^{-50}d^{-75}(d^{75}c^{-50})^{-2}.$$

$$d^{-75}(d^{75}c^{-50})^4d^{-75}(d^{75}c^{-50})^{-7}d^{-75}(d^{75}c^{-50})^4(d^{75}c^{-50})^{-2}d^{-50}.$$

$$c^{-6}d^{-50}c^{18}d^{-50}c^{-18}d^{-50}c^{48}d^{-50}c^{-18}d^{-50} = 1\rangle.$$

Theorem 3.3.5. *The direct product of any three of $PSL(2,\ p)$, p prime ≥ 5, $PSL(2,\ 25)$, $PSL(2,\ 27)$, $PSL(2,\ 49)$, $PSL(2,\ 81)$, $PSL(2,\ 169)$ is efficient.*

Proof. We can use Theorem 3.3.1 to construct efficient presentations on six generators for each of these direct products. In each case, it is easily checked that the group G given by this presentation is perfect and that the groups involved in the direct product display the properties required by Theorem 3.3.1.

$PSL(2,\ p) \times PSL(2,\ p) \times PSL(2,\ p)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(n\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(p\mp 1)/6}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^{-2}(xy^4xy^{(p+1)/2})^2(uv^4uv^{(p+1)/2})^2 = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ p) \times PSL(2,\ 25)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^2xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(p\mp 1)/6}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^4xy^{(p+1)/2})^2(uv^{-4}uv^3)^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ p) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^2xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(p\mp 1)/6}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^4xy^{(p+1)/2})^2(uv^{-3}uv^3)^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ p) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(p\mp 1)/6}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^4xy^{(p+1)/2})^2(u^{-1}v^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ p) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(p\mp 1)/6}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^4xy^{(p+1)/2})^2(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ p) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y^{\pm 1}(y^{(p\mp 1)/6}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm 1}(b^{(p\mp 1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^4xy^{(p+1)/2})^2(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 25) \times PSL(2,\ 25)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-4}xy^3)^2(u^7v^4u^7v^{(p+1)/2})^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 25) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^3ab^{-4})^2(xy^3xy^{-3})^2(uv^4uv^{(p+1)/2}uv^4u^{-17}v^{(p+1)/2})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 25) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^{-4}xy^3)^2(u^{-1}v^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 25) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^{-4}xy^3)^2(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 25) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^{-4}xy^3)^2(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 27) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm1}(v^{(p\mp1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2(xy^{-3}xy^3)^2(uv^4uv^{(p+1)/2}u^{-1}v^4u^{-3}v^{(p+1)/2})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 27) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^{-3}xy^3)^2(u^{-1}v^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 27) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^{-3}xy^3)^2(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 27) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{\pm1}(b^{(p\mp1)/6}xyu^{-1})^6 =$$

$$(ab^4ab^{(p+1)/2})^2(xy^{-3}xy^3)^2(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 49) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(u^3v^4u^3v^{(p+1)/2})^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 49) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^4uv^{(p+1)/2})^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 49) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}(u^{59}v^4u^{57}v^{(p+1)/2})^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 81) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{-1}(b^7uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{-1}(b^7xyu^{-1})^6 =$$

$$ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^4uv^{(p+1)/2})^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 81) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{-1}(b^7uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{-1}(b^7xyu^{-1})^6 =$$

$$ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}(uv^4u^{-1}v^{(p+1)/2})^{-2} = 1\rangle$$

$PSL(2,\ p) \times PSL(2,\ 169) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{\pm 1}(v^{(p\mp 1)/6}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^{14}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^{14}xyu^{-1})^6 =$$

$$ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{-5}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}.$$

$$(uv^4uv^{(p+1)/2}uv^4u^{-413}v^{(p+1)/2})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 25) \times PSL(2,\ 25)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^2xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-4}xy^3)^2(uv^{-4}uv^3)^{-2} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 25) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^2xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-4}xy^3)^2(uv^{-3}uv^3)^{-2} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 25) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-4}xy^3)^2(u^{-1}v^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 25) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-4}xy^3)^2(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 25) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-4}xy^3)^2(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^2xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^{-2}(xy^{-3}xy^3)^2(uv^{-3}uv^3)^2 = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-3}xy^3)^2(u^{-1}v^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-3}xy^3)^2(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 27) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2(xy^{-3}xy^3)^2(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 49) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 49) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 49) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 81) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^{-2}xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}.$$

$$uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 81) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^2xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSL(2,\ 169) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-4}ab^3)^{-2}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 27) \times PSL(2,\ 27)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^2xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2(xy^{-3}xy^3)^2(uv^{-3}uv^3)^{-2} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 27) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2(xy^{-3}xy^3)^2(u^{-1}v^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 27) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2(xy^{-3}xy^3)^2(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 27) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^2abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2(xy^{-3}xy^3)^2(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 49) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 49) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 49) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2x^{-1}y^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 81) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^{-2}xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}.$$

$$uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 81) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^2xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSL(2,\ 169) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^2uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^2xyu^{-1})^6 =$$

$$(ab^{-3}ab^3)^{-2}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 49) \times PSL(2,\ 49)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^4xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}.$$

$$(uv^2uv^{-2}uv^4uv^{-7}uv^4uv^{-2})^{-1} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 49) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$(ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2})^{-1}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}.$$

$$uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 49) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^4abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$(ab^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2})^{-1}xy^2xy^{-2}xy^4xy^{-7}xy^4xy^{-2}.$$

$$uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 81) \times PSL(2,\ 81)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$(a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2})^{-1}xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}.$$

$$uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4} = 1\rangle$$

$PSL(2,\ 49) \times PSL(2,\ 81) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2}xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}.$$

$$(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\,49) \times PSL(2,\,169) \times PSL(2,\,169)$:

$$G = \langle a,\, b,\, x,\, y,\, u,\, v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^4uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^4xyu^{-1})^6 =$$

$$(a^{-1}b^2ab^{-2}ab^4ab^{-7}ab^4ab^{-2})^{-1}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}.$$

$$uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5} = 1\rangle$$

$PSL(2,\,81) \times PSL(2,\,81) \times PSL(2,\,81)$:

$$G = \langle a,\, b,\, x,\, y,\, u,\, v \mid xy(xya^{-1})^2 = v^{-1}(v^7xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{-1}(b^7uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{-1}(b^7xyu^{-1})^6 =$$

$$(ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{-4})^{-1}xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}.$$

$$uv^2uv^{-4}uvuv^{-3}uv^5uv^{-3}uvuv^{-4} = 1\rangle$$

$PSL(2,\,81) \times PSL(2,\,81) \times PSL(2,\,169)$:

$$G = \langle a,\, b,\, x,\, y,\, u,\, v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{-1}(b^7uvx^{-1})^6 = ab(abu^{-1})^2 = y^{-1}(y^7abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{-1}(b^7xyu^{-1})^6 =$$

$$ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{-4}xy^2xy^{-4}xyxy^{-3}xy^5xy^{-3}xyxy^{-4}.$$

$$(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 81) \times PSL(2,\ 169) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b^{-1}(b^7uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b^{-1}(b^7xyu^{-1})^6 =$$

$$(ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{-4})^{-1}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}.$$

$$(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle$$

$PSL(2,\ 169) \times PSL(2,\ 169) \times PSL(2,\ 169)$:

$$G = \langle a,\ b,\ x,\ y,\ u,\ v \mid xy(xya^{-1})^2 = v(v^{14}xya^{-1})^6 =$$

$$uv(uvx^{-1})^2 = b(b^{14}uvx^{-1})^6 = ab(abu^{-1})^2 = y(y^{14}abu^{-1})^6 =$$

$$xy(xyu^{-1})^2 = b(b^{14}xyu^{-1})^6 =$$

$$(xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5})^{-1}xy^5xy^{-5}xy^6xy^{-8}xy^6xy^{-5}.$$

$$(uv^5uv^{-5}uv^6uv^{-8}uv^6uv^{-5})^{-1} = 1\rangle,$$

and so all of these direct products are efficient. $\square$

Of course we can use Lemma 3.3.2 to get efficient two generator presentations for all of these direct products. For example, putting $c = v^{14}xya^{-1}$, $d = b^2uvx^{-1}$ we obtain the following efficient presentation for $PSL(2,\ 27) \times PSL(2,\ 169)^2$ on two generators:

$$\langle c,\ d \mid c^{-255}d^{-6}(c^{-255}d^{-6}c^{-6}d^{26})^2 = d^{39}c^{-170}((d^{39}c^{-170})^{14}c^{-255}d^{-6}c^{-6}d^{26})^6$$

$$= c^{-170}(c^{-176}d^{26})^2 = d^{-6}(d^{-12}c^{-176}d^{26})^6$$

$$= (c^{-255}d^{18}c^{-255}d^{-18})^{-2}d^{-39}(d^{39}c^{-170})^5d^{-39}(d^{39}c^{-170})^{-5}d^{-39}(d^{39}c^{-170})^6.$$

$$d^{-39}(d^{39}c^{-170})^{-8}d^{-39}(d^{39}c^{-170})^6d^{-39}(d^{39}c^{-170})^{-5}d^{-26}c^{-24}d^{-26}c^{36}.$$

$$d^{-26}c^{-30}d^{-26}c^{54}d^{-26}c^{-30}d^{-26}c^{36} = 1\rangle.$$

Section 3.4. Efficient presentations for some direct products involving groups with trivial multiplier

Theorem 3.4.1. *Let G_1, G_2 be finite perfect groups, G_1 having multiplier C_2 and G_2 having trivial multiplier. Let G_1, G_2 have presentations of the form*

$$\langle a,\ b \mid a^{\alpha_1} = b^{\alpha_2} = (ab)^{\alpha_3} = w(a,\ b) = 1\rangle$$

$$\langle x,\ y \mid x^{\beta_1} = y^{\beta_2} = (xy)^{\beta_3} = v(x,\ y) = 1\rangle$$

with the α_i, β_i satisfying $\beta_3 \equiv \pm 1 \pmod{\alpha_1}$, $\beta_2 \equiv \pm 1 \pmod{\alpha_3}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_3 \equiv \pm 1 \pmod{\beta_1}$. Let the group G, given by the presentation below, be perfect:

$$\langle a,\ b,\ x,\ y \mid (xy)^{\pm 1}((xy)^{(\beta_3 \mp 1)/\alpha_1}a)^{\alpha_1} = y^{\pm 1}(y^{(\beta_2 \mp 1)/\alpha_3}ab)^{\alpha_3} =$$

$$(ab)^{\pm 1}((ab)^{(\alpha_3 \mp 1)/\beta_1}x)^{\beta_1} = b^{\pm 1}(b^{(\alpha_2 \mp 1)/\beta_1}x)^{\beta_1} = a^{-\alpha_1}w(a,\ b)v(x,\ y) = 1\rangle.$$

(The four congruences decide the choice of $\pm$ and $\mp$ in each of the four relations so that all of the powers within the relations are integer powers. In each relation one is chosen to be $+$ and the other $-$.)

Let G_1 be such that in the group presented by

$$\langle a,\ b \mid a^{\alpha_1} = s,\ b^{\alpha_2} = (ab)^{\alpha_3} = u,\ w(a,\ b) = t;$$

$$s,\ u,\ t \text{ central involutions}\rangle$$

we have $s = t$ and let G_2 be such that

$$\langle x,\ y \mid x^{\beta_1} = y^{\beta_2} = u,\ (xy)^{\beta_3} = s,\ v(x,\ y) = 1;$$

$$u,\ s \text{ central involutions}\rangle$$

is perfect. Then $G \cong G_1 \times G_2$ and so this direct product is efficient.

Proof. By Lemma 3.1.1, the following relations hold in G:

$$[ab,\ y] = [ab,\ x] = [b,\ x] = [a,\ xy] = 1,$$

$$b^{\alpha_2} = (ab)^{\alpha_3} = x^{-\beta_1} = y^{-\beta 2},\ a^{\alpha_1} = (xy)^{-\beta_3}.$$

Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$. From the relations $[ab,\ x] = 1$ and $[b,\ x] = 1$ we have $[a,\ x] = 1$. Similarly we have $[a,\ y] = 1$ and $[b,\ y] = 1$. Hence $[H,\ K] = 1$. Let $D = \langle x^{\beta_1},\ (xy)^{\beta_3},\ v(x,\ y)\rangle$. Clearly $D \leq H \cap K \leq Z(G)$. G is perfect with D central and $G/D \cong G_1 \times G_2$ so, by Lemma 1.5.1, D must be an epimorphic image of $M(G_1 \times G_2)$. Therefore D is either trivial or cyclic of order two. We also have that $G/D \cong G_1 \times G_2$, $H/D \cong G_1$ and $K/D \cong G_2$. Now in H the following relations hold:

$$a^{\alpha_1} = s,\ b^{\alpha_2} = (ab)^{\alpha_3} = u,\ w(a,\ b) = t,$$

with s, u, t, central involutions. By hypothesis we then have $s = t$ and so, by the fifth relation of G, $v(x,\ y) = 1$. Now consider K given by

$$\langle x,\ y \mid x^{\beta_1} = y^{\beta_2} = u,\ (xy)^{\beta_3} = s,\ v(x,\ y) = 1;\ u,\ s \text{ central involutions}\rangle.$$

By hypothesis this group is perfect and, since K is a central extension of G_2 by D, it is a stem extension of G_2 by D. We know that G_2 has trivial multiplier and so D must be trivial. Hence $G \cong G_1 \times G_2$ as required. □

Corollary 3.4.2. *The direct product, $G_1 \times G_2$, in Theorem 3.4.1 can be presented by*

$$\langle c,\ d \mid d^{\mp\alpha_3}c^{\mp\alpha_1}((d^{\mp\alpha_3}c^{\mp\alpha_1})^{(\beta_2\mp1)/\alpha_3}d^{\mp\beta_1})^{\alpha_1} =$$

$$c^{\mp\beta_3}d^{\mp\beta_1}((c^{\mp\beta_3}d^{\mp\beta_1})^{(\alpha_2\mp1)/\beta_1}d^{\pm\alpha_3})^{\beta_1} =$$

$$c^{\mp\beta_3\alpha_1}w(c^{\pm\beta_3},\ c^{\mp\beta_3}d^{\mp\beta_1})v(d^{\pm\alpha_3},\ d^{\mp\alpha_3}c^{\mp\alpha_1}) = 1\rangle$$

and so has an efficient presentation on two generators.

Proof. Put $c = (xy)^{(\beta_3\mp1)/\alpha_1}a$, $d = (ab)^{(\alpha_3\mp1)/\beta_1}x$ to get $a = c^{\pm\beta_3}$, $b = c^{\mp\beta_3}d^{\mp\beta_1}$, $x = d^{\pm\alpha_3}$, $y = d^{\mp\alpha_3}c^{\mp\alpha_1}$ and substitute into the presentation for G. □

In this section we will be using the following presentations as well as those given earlier:

$$SL(2,\ 8) = \langle x,\ y \mid x^2 = y^7 = (xy)^3 = (xy)^3(xy^{-3}xy^2xy^{-3})^2 = 1\rangle$$

$$SL(2,\ 16) = \langle x,\ y \mid x^2 = y^{15} = (xy)^3 = xy^3x^{-1}y^{-5}xy^3xy^{10} = 1\rangle$$

$$SL(2,\ 32) = \langle x,\ y \mid x^2 = y^{31} = (xy)^3 = xy^{28}xy^7xy^{-3}x^{-1}y^7 = 1\rangle$$

$$SL(2,\ 64) = \langle x,\ y \mid x^2 = y^{65} = (xy)^3 =$$

$$xy^{-1}xyxy^{-4}xy^5xy^{-5}xy^5xy^{-4}xyxy^{-1}x^{-1}y^4 = 1\rangle$$

$$PSL(3,\ 3) = \langle x,\ y \mid x^2 = y^{13} = (xy)^3 =$$

$$(y^4xy^{-1}x)^2y^2xy^{-3}xy^{-3}xy^2xy^{-1}x = 1\rangle$$

$$PSU(3,\ 3) = \langle x,\ y \mid x^3 = y^7 = (xy)^4 = xyx^{-1}y^{-2}xy^4xy^2 = 1\rangle$$

$$M_{11} = \langle x,\ y \mid x^2 = y^5 = (xy)^{11} = (xy^2)^2xy^{-1}xy^{-2}xy^2xy^{-1}xyxy^2xy^{-2} = 1\rangle$$

$$J_1 = \langle x,\ y \mid x^2 = y^3 = (xy)^{15} =$$

$$xy^{-1}xy(xyxy^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle.$$

The presentation for $SL(2,\ 8)$ comes from [2], and those for $SL(2,\ 16)$, $SL(2,\ 32)$ and $SL(2,\ 64)$ from [3]. The presentations for $PSL(3,\ 3)$ and $PSU(3,\ 3)$ come from [9], and those for M_{11} and J_1 from [14]. These groups all have trivial Schur multiplier [10].

Example 3.4.3. *$PSL(2,\ p) \times SL(2,\ 8)$, p prime ≥ 5, is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y(y^2ab)^3$$

$$= a^{-2}(ab^4a^{33}b^{(p+1)/2})^2(xy)^3xy^4xy^2(x^{-1}y^{-3})^2x^{-1}y^2x^{-1}y^{-3} = 1\rangle.$$

Proof. Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$. Then, using the same ideas as in earlier sections, the first four relations of G give us that $[H,\ K] = 1$ and that the following relations hold in G:

$$b^p = (ab)^3 = x^{-2} = y^{-7},\ a^2 = (xy)^{-3}.$$

Now let $D = \langle x^2,\ (xy)^3,\ (xy)^3xy^4xy^2(x^{-1}y^{-3})^2x^{-1}y^2x^{-1}y^{-3}\rangle$. Clearly we have $D \leq H \cap K \leq Z(G)$ and, since $G/D \cong PSL(2,\ p) \times SL(2,\ 8)$ which has trivial centre, we have $D = H \cap K = Z(G)$. We can show that G is perfect using matrices as follows. Let columns one to four represent x, y, a, b respectively.

$$\begin{pmatrix} 2 & -7 & 0 & 0 \\ 0 & 7 & 3 & 3 \\ 0 & 0 & -3 & p-3 \\ 3 & 3 & 2 & 0 \\ 1 & 2 & 66 & p+9 \end{pmatrix} \begin{matrix} row(4) - row(1) \\ row(1) - 2row(4) \\ \longrightarrow \\ row(5) - row(4) \end{matrix} \begin{pmatrix} 0 & -27 & -4 & 0 \\ 0 & 7 & 3 & 3 \\ 0 & 0 & -3 & p-3 \\ 1 & 10 & 2 & 0 \\ 0 & -8 & 64 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} -27 & -4 & 0 \\ 7 & 3 & 3 \\ 0 & -3 & p-3 \\ -8 & 64 & p+9 \end{pmatrix} \begin{matrix} row(1) - 4row(2) \\ row(2) - 7row(1) \\ \longrightarrow \\ row(4) - 8row(1) \end{matrix} \begin{pmatrix} 1 & 8 & 12 \\ 0 & -53 & -81 \\ 0 & -3 & p-3 \\ 0 & 128 & p+105 \end{pmatrix},$$

$$\begin{pmatrix} -53 & -81 \\ -3 & p-3 \\ 128 & p+105 \end{pmatrix} \begin{matrix} row(1) + 18row(2) \\ row(2) - 3row(1) \\ \longrightarrow \\ row(3) + 128row(1) \end{matrix} \begin{pmatrix} -1 & 18p-27 \\ 0 & -53p+78 \\ 0 & 2305p-3351 \end{pmatrix}.$$

For G to be perfect we require $53p-78$ and $2305p-3351$ to be coprime so that we can get 1 as a linear combination of the two. Now we can get $2305.78-3351.53 = 2187$ as a linear combination which is a power of three. Since $53p-78$ is obviously coprime to three we have G perfect.

Now since D is central and $G/D \cong PSL(2,\ p) \times SL(2,\ 8)$, D must be an epimorphic image of $M(PSL(2,\ p) \times SL(2,\ 8))$ and so is either trivial or isomorphic to C_2. Also we have $G/H \cong K/D \cong SL(2,\ 8)$ and $G/K \cong H/D \cong PSL(2,\ p)$. Now in H the following relations hold:

$$a^2 = s,\ b^p = (ab)^3 = u,\ a^{-2}(ab^4a^{33}b^{(p+1)/2})^2 = t,$$

with $s,\ u,\ t$ central involutions. We can use Lemma 3.1.3 to show that $s = t$ and so the fifth relation of G gives us that $(xy)^3xy^4xy^2(x^{-1}y^{-3})^2x^{-1}y^2x^{-1}y^{-3} = 1$. Now K is given by

$$\langle x,\ y \mid x^2 = y^7 = u,\ (xy)^3 = s,\ (xy)^3xy^4xy^2(x^{-1}y^{-3})^2x^{-1}y^2x^{-1}y^{-3} = 1;$$
$$u,\ s \text{ central involutions}\rangle.$$

We can easily check that K is perfect as follows:

$$\begin{pmatrix} 0 & 14 \\ 2 & -7 \\ 6 & 6 \\ 1 & 2 \end{pmatrix} \quad \xrightarrow[row(3)-6row(4)]{row(2)-2row(4)} \quad \begin{pmatrix} 0 & 14 \\ 0 & -11 \\ 0 & -6 \\ 1 & 2 \end{pmatrix}.$$

Since $\mathrm{hcf}(11,\ 14) = 1$ we can get a 1 in column two as required and K is perfect. So K is a perfect central extension of $SL(2,\ 8)$ which has trivial multiplier and so D must be trivial. Hence $G \cong PSL(2,\ p) \times SL(2,\ 8)$ as required and this direct product is efficient. □

Using Corollary 3.4.2 we can obtain an efficient presentation for $PSL(2,\ p) \times SL(2,\ 8)$ on two generators. Letting $c = xya$ and $d = abx$ we obtain:

$$PSL(2,\ p) \times SL(2,\ 8) = \langle c,\ d \mid c^{-3}d^{-2}((c^{-3}d^{-2})^{(p-1)/2}d^3)^2$$

$$= d^{-3}c^{-2}((d^{-3}c^{-2})^2d^{-2})^3 = c^{-6}(c^3(c^{-3}d^{-2})^4c^{99}(c^{-3}d^{-2})^{(p+1)/2})^2.$$

$$c^{-6}d^3(d^{-3}c^{-2})^4d^3(d^{-3}c^{-2})^2(d^{-3}(c^2d^3)^3)^2(d^{-3}c^{-2})^2d^{-3}(c^2d^3)^3 = 1\rangle.$$

Example 3.4.4. *$PSL(2,\ p) \times SL(2,\ 32)$, p prime ≥ 5, is efficiently presented by:*

$$G = \langle a,\ b,\ x,\ y \mid y(y^{15}a)^2 = ab(abx)^2 = b(b^{(p-1)/2}x)^2 = b^{\pm 1}(b^{(p\mp 1)/3}xy)^3$$

$$= a^{-2}(ab^4ab^{(p+1)/2})^2xy^{-3}xy^7xy^{28}x^{25}y^7 = 1\rangle.$$

Proof. In essence this proof works in the same way as that in Example 3.4.3. We use a modification of Theorem 3.4.1 similar to the modification made to Theorem 3.2.1 in Theorem 3.2.3. This is in order to facilitate the proof that G is perfect. Even so, in this case this perfectness proof is slightly more awkward. Let columns one to four correspond to the generators x, a, y, b respectively.

$$\begin{pmatrix} 0 & 2 & 31 & 0 \\ 2 & 3 & 0 & 3 \\ 2 & 0 & 0 & p \\ 1 & 2 & 3 & 0 \\ 28 & 2 & 39 & p+9 \end{pmatrix} \begin{array}{c} row(3)-row(2) \\ row(2)-2row(4) \\ \longrightarrow \\ row(5)-28row(4) \end{array} \begin{pmatrix} 0 & 2 & 31 & 0 \\ 0 & 3 & -6 & 3 \\ 0 & -3 & 0 & p-3 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & -45 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 31 & 0 \\ 3 & -6 & 3 \\ -3 & 0 & p-3 \end{pmatrix} \begin{array}{c} row(2)-row(1) \\ row(4)-row(1) \\ \longrightarrow \\ row(3)-3row(2) \end{array} \begin{pmatrix} 0 & 105 & 0 \\ 1 & -37 & 3 \\ 0 & -111 & p+6 \\ 0 & -76 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} 105 & 0 \\ -111 & p+6 \\ -76 & p+9 \end{pmatrix} \begin{array}{c} row(2)+row(1) \\ row(1)+17row(2) \\ row(2)+2row(1) \\ \longrightarrow \\ row(3)+25row(1) \\ row(1)+3row(3) \end{array} \begin{pmatrix} 0 & 1295p+7779 \\ 0 & 35p+210 \\ -1 & 426p+2559 \end{pmatrix}.$$

So, this group is perfect if and only if $1295p+7779$ and $35p+210$ are coprime. Since $5,\ 7 \mid 35p+210$, $5,\ 7 \mid 1295$ and 7779 is neither divisible by 5 or 7 the problem reduces to showing $1295p+7779$ and $p+6$ coprime. We can get $1.7779-6.1295=9$ as a linear combination of $1295p+7779$ and $p+6$ and so, since $p+6$ is obviously coprime to 3, we have G is perfect. $\square$

Example 3.4.5. *$PSL(2,\ p) \times PSU(3,\ 3)$, p prime ≥ 5, is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^3 = b^{\pm 1}(b^{(p \mp 1)/4}xy)^4 = xy(xyab)^3 = y(y^2ab)^3$$
$$= a^{-2}(ab^4ab^{(p+1)/2})^2x^{-1}y^5x^{-1}y^3x^{-1}y^2x^{-32}y^6 = 1\rangle.$$

Proof. In this case we use yet another modification of Theorem 3.4.1 making use of yet another set of congruences between the powers of a, b, ab, x, y, and xy. The mechanics of the proof are again the same. $\square$

We can also construct efficient direct products involving certain sporadic groups. We gives examples involving M_{11}, the Mathieu group of order 7920, and J_1, the Janko group of order 175560. Both of these groups have trivial Schur multiplier, [10].

Example 3.4.6. *$PSL(2,\ p) \times M_{11}$, p prime ≥ 5, is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$
$$= a^{-2}(a^{-1}b^4ab^{(p+1)/2})^2(xy^2)^2x^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^2x^{-1}y^{-1}x^{-1}yx^{-1}y^2x^{-1}y^{-2} = 1\rangle.$$

Proof. Following the proof of Theorem 3.4.1 with $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$, $D = \langle x^2,\ (xy)^{11},\ (xy^2)^2x^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^2x^{-1}y^{-1}x^{-1}yx^{-1}y^2x^{-1}y^{-2}\rangle$ we have that, in G, $[H,\ K] = 1$ and $a^2 = (xy)^{-11}$, $b^p = (ab)^3 = x^{-2} = y^{-5}$. Now we have $D = H \cap K = Z(G)$ and G is perfect as shown below (columns one to four representing $a,\ x,\ y,\ b$ respectively):

$$\begin{pmatrix} 2 & 11 & 11 & 0 \\ 0 & 2 & 0 & p \\ 3 & 2 & 0 & 3 \\ 0 & 2 & -5 & 0 \\ -2 & -5 & 3 & p+9 \end{pmatrix} \begin{array}{c} row(3) - row(1) \\ row(5) + row(1) \\ \longrightarrow \\ row(1) - 2row(3) \end{array} \begin{pmatrix} 0 & 29 & 33 & -6 \\ 0 & 2 & 0 & p \\ 1 & -9 & -11 & 3 \\ 0 & 2 & -5 & 0 \\ 0 & 6 & 14 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} 29 & 33 & -6 \\ 2 & 0 & p \\ 2 & -5 & 0 \\ 6 & 14 & p+9 \end{pmatrix} \begin{array}{c} row(2) - row(3) \\ row(1) - 14row(3) \\ row(4) - 5row(3) \\ \longrightarrow \\ row(3) - 2row(1) \\ row(4) + 4row(1) \end{array} \begin{pmatrix} 1 & 103 & -6 \\ 0 & 5 & p \\ 0 & -211 & 12 \\ 0 & 451 & p-15 \end{pmatrix},$$

$$\begin{pmatrix} 5 & p \\ -211 & 12 \\ 451 & p-15 \end{pmatrix} \begin{array}{c} row(2) + 42row(1) \\ row(3) - 92row(1) \\ \longrightarrow \\ row(1) + 5row(2) \\ row(3) - 9row(2) \end{array} \begin{pmatrix} 0 & 211p+60 \\ -1 & 42p+12 \\ 0 & -469p-123 \end{pmatrix}$$

Hence D is an epimorphic image of $M(PSL(2,\ p) \times M_{11})$ and so is either trivial or C_2. We also have that $G/H \cong K/D \cong M_{11}$ and $G/K \cong H/D \cong PSL(2,\ p)$.

Lemma 3.1.3 and the fifth relation of G give us that

$$(xy^2)^2x^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^2x^{-1}y^{-1}x^{-1}yx^{-1}y^2x^{-1}y^{-2} = 1.$$

It is easy to check that K is perfect and hence that D is trivial and $G \cong PSL(2,\ p) \times M_{11}$. $\square$

Example 3.4.7. *The direct product of $PSL(2,\ p)$, p prime ≥ 5, with J_1 is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^{(p-1)/2}x)^2 = b^{\pm1}(b^{(p\mp1)/3}y)^3$$
$$= a^2(ab^4a^{13}b^{(p+1)/2})^2xy^{-1}xy(xyx^3y^{-1})^3xy^{-1}xy(xy^{-1})^3.$$
$$xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle.$$

Proof. The proof mirrors that of the last example. The fact that G is perfect is illustrated below (columns one to four representing $a,\ x,\ y,\ b$ respectively):

$$\begin{pmatrix} 2 & 15 & 15 & 0 \\ 0 & 2 & 0 & p \\ 3 & 2 & 0 & 3 \\ 0 & 2 & -3 & 0 \\ 30 & 3 & -7 & p+9 \end{pmatrix} \quad \begin{array}{c} row(3) - row(1) \\ row(5) - 15row(1) \\ \longrightarrow \\ row(1) - 2row(3) \end{array} \quad \begin{pmatrix} 0 & 41 & 45 & 0 \\ 0 & 2 & 0 & p \\ 1 & -13 & -15 & 3 \\ 0 & 2 & -3 & 0 \\ 0 & -222 & -232 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} 41 & 45 & 0 \\ 2 & 0 & p \\ 2 & -3 & 0 \\ -222 & -232 & p+9 \end{pmatrix} \xrightarrow[\substack{row(1)-20row(3) \\ row(3)-2row(1) \\ row(4)-9row(1)}]{\substack{row(4)+7row(1) \\ row(2)-row(3) \\ row(4)-28row(3)}} \begin{pmatrix} 1 & 105 & 0 \\ 0 & 3 & p \\ 0 & -213 & 0 \\ 0 & -778 & p+9 \end{pmatrix},$$

$$\begin{pmatrix} 3 & p \\ -213 & 0 \\ -778 & p+9 \end{pmatrix} \xrightarrow[row(1)+3row(3)]{\substack{row(2)+71row(1) \\ row(3)+259row(1)}} \begin{pmatrix} 0 & 781p+27 \\ 0 & 71p \\ -1 & 260p+9 \end{pmatrix}.$$

For G perfect it is enough to show then that $781p+27$ and $71p$ are coprime. This is obviously the case for all $p \geq 5$ since $71p$ has only 71 and p as factors and since $71 \mid 781$ neither of these divides $781p+27$. $\square$

Theorem 3.4.8. *The direct product of any one of $PSL(2,\ p)$, p prime ≥ 5, $PSL(2,\ 25)$, $PSL(2,\ 27)$, $PSL(2,\ 49)$, $PSL(2,\ 81)$, $PSL(2,\ 169)$ with any one of $SL(2,\ 8)$, $SL(2,\ 16)$, $SL(2,\ 32)$, $SL(2,\ 64)$, $PSL(3,\ 3)$, $PSU(3,\ 3)$, M_{11}, J_1 is efficient.*

Proof. We use Theorem 3.4.1 for the majority of the direct products. Those involving $PSL(2,\ 16)$ make use of the modification of the type in Example 3.4.4 and those involving $PSU(3,\ 3)$ make use of the modification of the type in Example 3.4.5. We also use a slight modification in the direct products involving $PSL(2,\ 81)$ and $PSL(2,\ 169)$. We follow Theorem 3.4.1 but make use of the fact that $u = t$ rather than $s = t$ in H. The four generator presentations are given below:

$PSL(2,\ p) \times SL(2,\ 8)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y(y^2ab)^3 =$$
$$a^{-2}(ab^4a^{33}b^{(p+1)/2})^2(xy)^3xy^4xy^2(x^{-1}y^{-3})^2x^{-1}y^2x^{-1}y^{-3} = 1\rangle.$$

$PSL(2,\ p) \times SL(2,\ 16)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^7a)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = b^{p\pm1}(b^{(p\mp1)/3}xy)^3$$
$$= a^{-2}(ab^4ab^{(p+1)/2})^2xy^{-3}xy^5xy^{-3}x^{19}y^{-10} = 1\rangle$$

$PSL(2,\ p) \times SL(2,\ 32)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^{15}a)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = b^{p\pm1}(b^{(p\mp1)/3}xy)^3$$
$$= a^{-2}(ab^4ab^{(p+1)/2})^2xy^{-3}xy^7xy^{28}x^{25}y^7 = 1\rangle$$

$PSL(2,\ p) \times SL(2,\ 64)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^{32}a)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = b^{p\pm1}(b^{(p\mp1)/3}xy)^3$$
$$= a^2(ab^4ab^{(p+1)/2})^2xy^4xy^{-1}xyxy^{-4}xy^5xy^{-5}xy^5xy^{-4}xyx^{-25}y^{-1} = 1\rangle$$

$PSL(2,\ p) \times PSL(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^6a)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = b^{p\pm 1}(b^{(p\mp 1)/3}xy)^3$$

$$= a^{-2}(ab^4ab^{(p+1)/2})^2xy^2xy^{10}xy^{-3}xy^2x^{11}y^{-1}(xy^4xy^{-1})^2 = 1\rangle$$

$PSL(2,\ p) \times PSU(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^2 = b^{p\pm 1}(b^{(p\mp 1)/4}ab)^4 = xy(xyab)^3 = y(y^2ab)^3$$

$$= a^{-2}(a^{-1}b^4ab^{(p+1)/2})^2xy^2xy^{10}xy^{-3}xy^2x^{11}y^{-1}(xy^4xy^{-1})^2 = 1\rangle$$

$PSL(2,\ p) \times M_{11}$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^{(p-1)/2}x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$

$$= a^{-2}(a^{-1}b^4ab^{(p+1)/2})^2(xy^2)^2x^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^2x^{-1}y^{-1}x^{-1}yx^{-1}y^2x^{-1}y^{-2} = 1\rangle$$

$PSL(2,\ p) \times J_1$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^{(p-1)/2}x)^2 = b^{\pm 1}(b^{(p\mp 1)/3}y)^3$$

$$= a^2(ab^4a^{13}b^{(p+1)/2})^2.$$

$$xy^{-1}xy(xyx^3y^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle$$

$PSL(2,\ 25) \times SL(2,\ 8)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^2ab)^3$$

$$= a^{-2}(ab^3ab^{-4})^2(xy)^3(xy^{-3}xy^2xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 25) \times SL(2,\ 16)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^7a)^2 = b(b^6x)^2 = ab(abx)^2 = b(b^4xy)^3$$

$$= a^{-2}(ab^3ab^{-4})^2(xy^3x^{-1}y^{-5}xy^3xy^{10})^{-1} = 1\rangle$$

$PSL(2,\ 25) \times SL(2,\ 32)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^{10}ab)^3$$

$$= a^{-2}(ab^3ab^{-4})^2(xy)^3(xy^{-3}xy^7)^2 = 1\rangle$$

$PSL(2,\ 25) \times SL(2,\ 64)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y^{-1}(y^{22}ab)^3$$

$$= a^{-2}(ab^3ab^{-4})^2(xy)^3xy^{-1}xyxy^{-4}xy^5xy^{-5}xy^5xy^{-4}xyxy^{-1}xy^4 = 1\rangle$$

$PSL(2,\ 25) \times PSL(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= a^{-2}(ab^3ab^{-4})^2(xy^4xy^{-1})^2xy^2(xy^{-3})^2xy^2xy^{-1} = 1\rangle$$

$PSL(2,\ 25) \times PSU(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^2 = b(b^3ab)^4 = xy(xyab)^3 = y(y^2ab)^3$$

$$= a^{-2}(ab^3ab^{-4})^2xyx^{-1}y^{-2}xy^4xy^2 = 1\rangle$$

$PSL(2,\ 25) \times M_{11}$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^6x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$

$$= a^{-2}(ab^3ab^{-4})^2(xy^2)^2xy^{-1}xy^{-2}xy^2xy^{-1}xyxy^2xy^{-2} = 1\rangle$$

$PSL(2,\ 25) \times J_1$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^6x)^2 = b(b^4y)^3$$

$$= a^{-2}(ab^3ab^{-4})^2xy^{-1}xy(xyxy^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle$$

$PSL(2,\ 27) \times SL(2,\ 8)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^2ab)^3$$

$$= a^{-2}(ab^3ab^{-3})^2(xy)^3(xy^{-3}xy^2xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 27) \times SL(2,\ 16)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^7a)^2 = b(b^6x)^2 = ab(abx)^2 = b(b^4xy)^3$$

$$= a^{-2}(ab^3ab^{-3})^2xy^3x^{-1}y^{-5}xy^3xy^{10} = 1\rangle$$

$PSL(2,\ 27) \times SL(2,\ 32)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^{10}ab)^3$$

$$= a^{-2}(ab^3ab^{-3})^2(xy)^3(xy^{-3}xy^7)^2 = 1\rangle$$

$PSL(2,\ 27) \times SL(2,\ 64)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y^{-1}(y^{22}ab)^3$$

$$= a^{-2}(ab^3ab^{-3})^2(xy)^3xy^{-1}xyxy^{-4}xy^5xy^{-5}xy^5xy^{-4}xyxy^{-1}xy^4 = 1\rangle$$

$PSL(2,\ 27) \times PSL(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^6x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= a^{-2}(a^{-1}b^3ab^{-3})^2(xy^4xy^{-1})^2xy^2(xy^{-3})^2xy^2xy^{-1} = 1\rangle$$

$PSL(2,\ 27) \times PSU(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^2 = b(b^3ab)^4 = xy(xyab)^3 = y(y^2ab)^3$$

$$= a^{-2}(ab^3ab^{-3})^2xyx^{-1}y^{-2}xy^4xy^2 = 1\rangle$$

$PSL(2,\ 27) \times M_{11}$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^6x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$

$$= a^{-2}(ab^3ab^{-3})^2(xy^2)^2xy^{-1}xy^{-2}xy^2xy^{-1}xyxy^2xy^{-2} = 1\rangle$$

$PSL(2,\ 27) \times J_1$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^6x)^2 = b(b^4y)^3$$

$$= a^{-2}(ab^3ab^{-3})^2xy^{-1}xy(xyxy^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle$$

$PSL(2,\ 49) \times SL(2,\ 8)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{12}x)^2 = ab(abx)^2 = y(y^2ab)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}(xy)^3(xy^{-3}xy^2xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 49) \times SL(2,\ 16)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^7a)^2 = b(b^{12}x)^2 = ab(abx)^2 = b(b^8xy)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}xy^3x^{-1}y^{-5}xy^3xy^{10} = 1\rangle$$

$PSL(2,\ 49) \times SL(2,\ 32)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{12}x)^2 = ab(abx)^2 = y(y^{10}ab)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}(xy)^3(xy^{-3}xy^7)^2 = 1\rangle$$

$PSL(2,\ 49) \times SL(2,\ 64)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{12}x)^2 = ab(abx)^2 = y^{-1}(y^{22}ab)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}(xy)^3xy^{-1}xyxy^{-4}xy^5xy^{-5}xy^5xy^{-4}xyxy^{-1}xy^4 = 1\rangle$$

$PSL(2,\ 49) \times PSL(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{12}x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}(xy^4xy^{-1})^2x^{-1}y^2(xy^{-3})^2xy^2xy^{-1} = 1\rangle$$

$PSL(2,\ 49) \times PSU(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^2 = b(b^6ab)^4 = xy(xyab)^3 = y(y^2ab)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}xyx^{-1}y^{-2}xy^4xy^2 = 1\rangle$$

$PSL(2,\ 49) \times M_{11}$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^{12}x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}(xy^2)^2xy^{-1}xy^{-2}xy^2xy^{-1}xyxy^2xy^{-2} = 1\rangle$$

$PSL(2,\ 49) \times J_1$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^{12}x)^2 = b(b^8y)^3$$

$$= a^{-1}b^2ab^{23}ab^4ab^{18}ab^4ab^{23}.$$

$$xy^{-1}xy(xyx^3y^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle$$

$PSL(2,\ 81) \times SL(2,\ 8)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{20}x)^2 = ab(abx)^2 = y(y^2ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}(xy)^3(xy^{-3}xy^2xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 81) \times SL(2,\ 16)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^7a)^2 = b(b^{20}x)^2 = ab(abx)^2 = b^{-1}(b^{14}xy)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}xy^3x^{-1}y^{-5}xy^3xy^{10} = 1\rangle$$

$PSL(2,\ 81) \times SL(2,\ 32)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{20}x)^2 = ab(abx)^2 = y(y^{10}ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{-37}(xy)^3(xy^{-3}xy^7)^2 = 1\rangle$$

$PSL(2,\ 81) \times SL(2,\ 64)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{20}x)^2 = ab(abx)^2 = y^{-1}(y^{22}ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}(xy)^3xy^{-1}xyxy^{-4}.$$

$$xy^5xy^{-5}xy^5xy^{-4}xyxy^{-1}xy^4 = 1\rangle$$

$PSL(2,\ 81) \times PSL(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{20}x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}(xy^4xy^{-1})^2xy^2(xy^{-3})^2xy^2xy^{-1} = 1\rangle$$

$PSL(2,\ 81) \times PSU(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^2 = b(b^{10}ab)^4 = xy(xyab)^3 = y(y^2ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}xyx^{-1}y^{-2}xy^4xy^2 = 1\rangle$$

$PSL(2,\ 81) \times M_{11}$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^{20}x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}(xy^2)^2xy^{-1}xy^{-2}xy^2xy^{-1}xyxy^2xy^{-2} = 1\rangle$$

$PSL(2,\ 81) \times J_1$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^{20}x)^2 = b^{-1}(b^{14}y)^3$$

$$= ab^2ab^{-4}abab^{-3}ab^5ab^{-3}abab^{37}.$$

$$xy^{-1}xy(xyxy^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle$$

$PSL(2,\ 169) \times SL(2,\ 8)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{42}x)^2 = ab(abx)^2 = y(y^2ab)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}(xy)^3(xy^{-3}xy^2xy^{-3})^2 = 1\rangle$$

$PSL(2,\ 169) \times SL(2,\ 16)$:

$$G = \langle a,\ b,\ x,\ y \mid y(y^7a)^2 = b(b^{42}x)^2 = ab(abx)^2 = b(b^{28}xy)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}xy^3x^{-1}y^{-5}xy^3xy^{10} = 1\rangle$$

$PSL(2,\ 169) \times SL(2,\ 32)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{42}x)^2 = ab(abx)^2 = y(y^{10}ab)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}(xy)^3(xy^{-3}xy^7)^2 = 1\rangle$$

$PSL(2,\ 169) \times SL(2,\ 64)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{42}x)^2 = ab(abx)^2 = y^{-1}(y^{22}ab)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}(xy)^3xy^{-1}xyxy^{-4}xy^5xy^{-5}xy^5xy^{-4}xyxy^{-1}xy^4 = 1\rangle$$

$PSL(2,\ 169) \times PSL(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid xy(xya)^2 = b(b^{42}x)^2 = ab(abx)^2 = y(y^4ab)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}(xy^4xy^{-1})^2xy^2(xy^{-3})^2xy^2xy^{-1} = 1\rangle$$

$PSL(2,\ 169) \times PSU(3,\ 3)$:

$$G = \langle a,\ b,\ x,\ y \mid x(xa)^2 = b(b^{21}ab)^4 = xy(xyab)^3 = y(y^2ab)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}xyx^{-1}y^{-2}xy^4xy^2 = 1\rangle$$

$PSL(2,\ 169) \times M_{11}$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^5a)^2 = b(b^{42}x)^2 = ab(abx)^2 = y^{-1}(y^2ab)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}(xy^2)^2xy^{-1}xy^{-2}xy^2xy^{-1}xyxy^2xy^{-2} = 1\rangle$$

$PSL(2,\ 169) \times J_1$:

$$G = \langle a,\ b,\ x,\ y \mid xy((xy)^7a)^2 = ab(abx)^2 = b(b^{42}x)^2 = b(b^{28}y)^3$$

$$= ab^5ab^{-5}ab^6ab^{-8}ab^6ab^{80}.$$

$$xy^{-1}xy(xyxy^{-1})^3xy^{-1}xy(xy^{-1})^3xyxy^{-1}(xy)^3(xy^{-1}(xy)^2)^{-7} = 1\rangle.$$

Hence all of these direct products are efficient. □

Section 3.5. Other related results

Theorem 3.5.1. *$PSL(2,\ 5)^2 \times SL(2,\ 5)$ is efficient with presentation:*

$$G = \langle a,\ b,\ x,\ y \mid b(bxy^{-1})^4 = a^2 = y(y^2a)^2 = x(xa)^2 = (ab)^{-1}((ab)^2x)^2$$

$$= ab^{-2}ab^4ab^{-2}ababxyx^{-1}y^2x^{-1}y = 1\rangle.$$

Proof. We use the following presentations:

$$PSL(2,\ 5)^2 = \langle a,\ b \mid a^2 = b^{15} = (ab)^5 = ab^{-2}ab^4ab^{-2}abab = 1\rangle$$

$$SL(2,\ 5) = \langle x,\ y \mid x^3 = y^5 = (xy^{-1})^4 = xyx^{-1}y^2x^{-1}y = 1\rangle.$$

The presentation for $PSL(2,\ 5)^2$ comes from [14]. The presentation for $SL(2,\ 5)$ is obtained by using the program given in Section 2.3. Let $H = \langle a,\ b\rangle$, $K = \langle x,\ y\rangle$. Then $[H,\ K] = 1$ and the following relations hold in G:

$$b^{15} = (xy^{-1})^{-4},\ y^5 = a^{-2} = x^3 = (ab)^{-5}.$$

We can check that G is perfect using matrix methods or by computer. Let $D = \langle (xy^{-1})^4,\ xyx^{-1}y^2x^{-1}y\rangle$. Then $G/D \cong PSL(2,\ 5)^2 \times SL(2,\ 5)$, $D \leq H \cap K \leq Z(G)$, and, since G is perfect, D is an epimorphic image of $C_2 \times C_2$.

$$K = \langle x,\ y \mid x^3 = y^5 = 1,\ (xy^{-1})^4 = u,\ xyx^{-1}y^2x^{-1}y = t;$$

$$u,\ t \text{ central involutions}\rangle.$$

This group is perfect and so must be $SL(2,\ 5)$. Hence D is trivial and $G = PSL(2,\ 5)^2 \times SL(2,\ 5)$. $\square$

We can use this presentation to get an efficient presentation for $PSL(2,\ 5)^2 \times SL(2,\ 5)$ on two generators. Let $c = bxy^{-1}$, $d = y^2a$. Then $b = c^{-4}$, $xy^{-1} = c^5$, $y = d^{-2}$, $a = d^5$ and so $x = c^5d^{-2}$. Substituting these into the four generator presentation yields:

$$\langle c,\ d \mid d^{10} = c^5d^{-2}(c^5d^3)^2 = cd^3c^{-4}d^5cd^3 =$$

$$c^8d^5c^{-16}d^5c^8d^5c^{-4}d^5cd^{-2}c^{-5}d^{-2}c^{-5}d^3 = 1\rangle.$$

Theorem 3.5.2. *$PSL(2,\ p) \times C_n$, p prime ≥ 5 is efficient when any two of* *the following hold:*

$(i)\quad (n,\ 2) = 1$

$(ii)\quad (n,\ 3) = 1$

(iii) *Either* $n \equiv \pm 1 \pmod p$ *or* $p \equiv \pm 1 \pmod n$.

Proof. If (i) and (ii) hold (i.e. $(n,\ 6) = 1$) consider

$$G = \langle a,\ b,\ x \mid x(x^{(n-1)/2}a)^2 = x^{\pm 1}(x^{(n\mp 1)/3}ab)^3 = a^{-2}b^p = (ab^4ab^{(p+1)/2})^2 = 1\rangle.$$

The first two relations, by Lemma 3.1.1, give us that $[a,\ x] = [b,\ x] = 1$ and that $a^2 = (ab)^3 = x^{-n}$. So,

$$G = \langle a,\ b,\ x \mid a^2 = b^p = (ab)^3 = x^{-n} = s,\ (ab^4ab^{(p+1)/2})^2 = 1;$$
$$s \text{ central}\rangle.$$

If (iii) holds and $n \equiv \pm 1 \pmod p$ then let R be the word $x^{\pm 1}(x^{(n\mp 1)/p}b)^p$. If (iii) holds and $p \equiv \pm 1 \pmod n$ then let R be the word $b^{\pm 1}(b^{(p\mp 1)/n}x)^n$. Consider the case where (i) and (iii) hold and let G be the group defined by the presentation:

$$\langle a,\ b,\ x \mid x(x^{(n-1)/2}a)^2 = R = a^{-2}(ab)^3 = (ab^4ab^{(p+1)/2})^2 = 1\rangle.$$

Lemma 3.1.1 gives us that $[a,\ x] = [b,\ x] = 1$ and $a^2 = b^p = x^{-n}$ and so again

$$G = \langle a,\ b,\ x \mid a^2 = b^p = (ab)^3 = x^{-n} = s,\ (ab^4ab^{(p+1)/2})^2 = 1;$$
$$s \text{ central}\rangle.$$

Now consider the final case where (ii) and (iii) hold. Let G be the group defined by the presentation:

$$\langle a,\ b,\ x \mid x^{\pm 1}(x^{(n\mp 1)/3}ab)^3 = R = a^{-2}b^p = (ab^4ab^{(p+1)/2})^2 = 1\rangle.$$

This time Lemma 3.1.1 gives us that $[a,\ b] = [a,\ x] = 1$ and $b^p = (ab)^3 = x^{-n}$ and so in each case we have

$$G = \langle a,\ b,\ x \mid a^2 = b^p = (ab)^3 = x^{-n} = s,\ (ab^4ab^{(p+1)/2})^2 = 1;$$
$$s \text{ central}\rangle.$$

Consider the subgroup of this group, G, generated by a and b:

$$H = \langle a,\ b \mid a^2 = b^p = (ab)^3 = s, (ab^4ab^{(p+1)/2})^2 = 1;$$
$$s \text{ central}\rangle.$$

It is easy to see that this group is perfect:

$$\begin{pmatrix} 2 & -p \\ 1 & 3 \\ 4 & p+9 \end{pmatrix} \quad \begin{matrix} row(1) - 2row(2) \\ \longrightarrow \\ row(4) - 4row(2) \end{matrix} \quad \begin{pmatrix} 0 & -p-6 \\ 1 & 3 \\ 0 & p-3 \end{pmatrix}.$$

We need to be able to get 1 as a linear combination of $p+6$ and $p-3$. We can obviously get 9 and since $p-3$ is not divisible by 3 the result follows. So, H is a perfect stem extension of $PSL(2,\ p)$. Lemma 3.1.3 gives us that s must be trivial and so, in each of the three cases we have an efficient presentation of the direct product $PSL(2,\ p) \times C_n$. □

We can also get efficient presentations for these direct products on two generators using ideas similar to those used in earlier sections.

Example. Putting $c = xab$, $d = b$, we can obtain an efficient presentation for $PSL(2,\ p) \times C_2$, p prime ≥ 5, on two generators:

$$PSL(2,\ p) \times C_2 = \langle c,\ d \mid d(d^{(p-1)/2}c^3)^2 = (dc^2)^{-2}d^p$$

$$= (c^{-2}d^3c^{-2}d^{(p+1)/2})^2 = 1\rangle.$$

Chapter 4. Inefficient groups

In his paper of 1964, [26], Swan introduced a class of groups which he proved to be inefficient. The groups are described by the presentations:

$$Sw_n = \langle a_1,\ a_2,\ \dots, a_n, c \mid c^3 = 1,\ c^{-1}a_i c = a_i^2,$$

$$[a_i,\ a_j] = 1,\ 1 \leq i,\ j \leq n\rangle,$$

and are inefficient for $n \geq 3$. Sw_n has order 3.7^n and is the split extension of a direct product of n copies of C_7 by C_3. Swan proved that they have trivial Schur multiplier and deficiency tending to infinity with n.

In [30], Wiegold generalises the Swan groups. He constructs 3-generator inefficient groups.

More recently, Wotherspoon, [31], proves that a class of groups originally introduced by Coxeter are inefficient subject to certain conditions.

In [18], Kovàcs gives us a class of inefficient perfect groups, the direct powers of which are also inefficient.

Section 4.1. A new class of inefficient groups

Theorem 4.1.1. *The group given by the presentation:*

$$G_n = \langle a,\ b,\ c,\ d \mid a^2 = b^n = c^n = d^n = (ab)^2 = (ac)^2 = [a,\ d] = [b,\ d]$$
$$= [c,\ d] = [b,\ c]d^{-1} = 1\rangle$$

is inefficient whenever $n \geq 3$ is odd.

Proof. We prove this via a series of lemmas.

Lemma 4.1.2. *G_n has order $2n^3$.*

Proof. Consider the group given by:

$$H_n = \langle b,\ c,\ d \mid b^n = c^n = d^n = [b,\ d] = [c,\ d] = [b,\ c]d^{-1} = 1\rangle,\ n \text{ odd}.$$

We can check that G_n is the semidirect product of H_n by $\langle a\rangle$. In the presentation for G_n we can see that a leaves d fixed and inverts b and c. The only work we have to do to show that G_n is the semidirect product of H_n by $\langle a\rangle$ is in showing that $[b^{-1},\ c^{-1}]d^{-1} = 1$. This follows easily since d is central. So G_n has order twice that of H_n (we can rule out the possibility that a is trivial since it has order 2 in the abelianisation of G). Now let K_n be the subgroup of H_n generated by c

and d. It is not difficult to check that H_n is the semidirect product of

$$K_n = \langle c,\ d \mid c^n = d^n = [c,\ d] = 1 \rangle$$

by $\langle b \rangle$. K_n is obviously the direct product $C_n \times C_n$ and so d has order n in K_n. So d must have order exactly n in H_n. The element d of H_n is obviously central and so we can factor out by $\langle d \rangle$ to get:

$$H_n/\langle d \rangle = \langle b,\ c \mid b^n = c^n = [b,\ c] = 1 \rangle$$

which is obviously the direct product $C_n \times C_n$. So, G_n has the structure:

$$\begin{array}{l} \bullet \\ | \ \} \ C_2 \\ \bullet \\ | \ \} \ C_n \times C_n \\ \bullet \\ | \ \} \ C_n \\ \bullet \end{array}$$

and so has order $2n^3$. $\square$

Lemma 4.1.3. *Any group with trivial Schur multiplier having a subgroup of index two with multiplier of rank greater than or equal to two is inefficient.*

Proof. Consider, generally, a finite group G given by a presentation on g generators with r relators. Let H be a subgroup of index k in G. Following the Reidemeister-Schreier method for obtaining a presentation of this subgroup we would get a presentation on $(g-1)k+1$ generators with rk relators and so the deficiency would be $(r-g+1)k-1$. If the deficiency of the presentation for G were d then that for the subgroup would be $(d+1)k-1$.

In our case, if we take $d=0$, $k=2$ we get a presentation for our subgroup of index two of deficiency one. This gives a contradiction, since our subgroup has multiplier of rank two, and so G cannot have a zero deficiency presentation. Hence G is inefficient. $\square$

Lemma 4.1.4. *G_n has trivial Schur multiplier.*

Proof. Lemma 1.5.2 gives us that the exponent of $M(G)$ divides the order of G for a group G. Also Lemma 1.5.3 gives us that $M(G)_k \leq M(G_k)$, k prime. Putting $k=2$ then gives us that the multiplier of G_n has no factor C_2. So, the multiplier of G_n must contain only factors of the form C_m and C_{m^2} where m divides n. Now if the multiplier of G_n is non-trivial there must exist a stem extension of G_n of the form:

$$\bar{G_n} = \langle a,\ b,\ c,\ d,\ z \mid a^2 = z^p,\ b^n = z^q,\ c^n = z^r,\ d^n = z^s,$$

$$(ab)^2 = z^t,\ (ac)^2 = z^u,\ [a,\ d] = z^v,\ [b,\ d] = z^w,\ [c,\ d] = z^x,$$

$$[b,\ c]d^{-1} = z^y,\ z^n = 1;\ z \text{ central}\rangle.$$

Firstly, consider the relation $a^2 = z^p$. Since n is odd we can find an m such that $2m \equiv p \pmod{n}$. Choose $\alpha = az^{-m}$. Now,

$$a^2 = z^p \Leftrightarrow \alpha^2 z^{2m} = z^p \Leftrightarrow \alpha^2 = 1$$

and so, without loss of generality, we can assume that $a^2 = 1$.

Next, consider the relation $(ab)^2 = z^t$. Since n is odd we can find an m and hence a $\beta = bz^{-m}$ such that $2m \equiv t \pmod{n}$. Now,

$$(ab)^2 = z^t \Leftrightarrow a\beta z^m a\beta z^m = z^t \Leftrightarrow (a\beta)^2 = 1$$

and so, without loss of generality, we can assume that $(ab)^2 = 1$ and similarly that $(ac)^2 = 1$.

Now, using the relation $b^n = z^q$ and $(ab)^2 = 1$ we have:

$$z^{-q} = b^{-n} = a^{-1}b^n a = a^{-1}z^q a = z^q$$

and so $z^{2q} = 1$. Since z has odd order we must have $z^q = 1$ and so $b^n = 1$. Similarly $c^n = 1$.

Now consider the three relations $[a,\ d] = z^v$, $[b,\ d] = z^w$, $[c,\ d] = z^x$. Write these as

$$(1)\quad a = d^{-1}adz^{-v},$$

$$(2)\quad b = d^{-1}bdz^{-w},$$

$$(3)\quad c = d^{-1}cdz^{-x}.$$

The relation $a^2 = 1$ and (1) give us that $z^{-2v} = 1$ and, since the order of z is odd, we must have $z^v = 1$. Now the relation $(ab)^2 = 1$, (1) and (2) give us that $z^{-2v-2w} = z^{-2w} = 1$ and so $z^w = 1$. Similarly the relation $(ac)^2 = 1$, (1) and (3) give us that $z^x = 1$. So, we have that $[a,\ d] = [b,\ d] = [c,\ d] = 1$.

Now consider the relation $[b,\ c]d^{-1} = z^y \Leftrightarrow b^{-1}c^{-1}b = c^{-1}dz^y$. Taking each side of this relation to the power n we obtain:

$$1 = b^{-1}c^{-n}b = (c^{-1}dz^y)^n = c^{-n}d^nz^{ny} = d^n$$

and so $d^n = 1$.

Now we are left with:

$$\bar{G}_n = \langle a,\ b,\ c,\ d,\ z \mid a^2 = b^n = c^n = d^n = (ab)^2 = (ac)^2 = [a,\ d] = [b,\ d]$$
$$= [c,\ d] = 1,\ [b,\ c]d^{-1} = z^y,\ z^n = 1;\ z \text{ central}\rangle.$$

If this is a stem extension of G_n we require $z \in G_n'$. So, we require z to be trivial in the abelianisation of $\bar{G}_n$. Consider the matrix corresponding to $\bar{G}_n/\bar{G}_n'$, columns one to five representing a, b, c, d, z respectively.

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 \\ 0 & 0 & n & 0 & 0 \\ 0 & 0 & 0 & n & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & n \end{pmatrix} \begin{array}{c} row(5) - row(1) \\ row(6) - row(1) \\ row(2) - (n-1)/2row(5) \\ row(5) - 2row(2) \\ \longrightarrow \\ row(3) - (n-1)/2row(6) \\ row(6) - 2row(3) \\ row(4) - nrow(7) \\ row(4) + yrow(8) \end{array} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 & n \end{pmatrix}.$$

For z to be trivial in $\bar{G}_n/\bar{G}'_n$ we need the determinant of the matrix formed by removing the zero rows from the matrix on the right to be equal to ± 2. This is obviously not the case unless $n = \pm 1$. So no such non-trivial stem extension exists and the multiplier of G_n is trivial. $\square$

Lemma 4.1.5. *The index two subgroup of G_n with presentation:*

$$H_n = \langle b,\ c,\ d \mid b^n = c^n = d^n = [b,\ d] = [c,\ d] = [b,\ c]d^{-1} = 1\rangle$$

has Schur multiplier of rank at least two.

Proof. Consider the group given by the presentation:

$$\bar{H}_n = \langle b,\ c,\ d \mid b^n = c^n = d^n = z^n = t^n = [b,\ c]d^{-1} = 1,$$

$$[b,\ d] = z,\ [c,\ d] = t;\ z,\ t \text{ central}\rangle.$$

$\bar{H}_n$ is obviously a stem extension of H_n since z and t are both central and obviously trivial in the abelianisation of $\bar{H}_n$ due to the relations $[b,\ d] = z,\ [c,\ d] = t$. Consider the following permutations on n^2 symbols:

$$P_1 = (1\ \ 1+n\ \ 1+2n\ ...\ 1+(n-1)n)(2\ \ 2+n\ \ 2+2n\ ...\ 2+(n-1)n)...$$

$$...(n\ \ 2n\ \ 3n\ ...\ n^2),$$

$$P_2 = (1\ \ 2\ \ 3\ ...\ n)^{1.2/2}(1+n\ \ 2+n\ \ 3+n\ ...\ 2n)^{2.3/2}...$$

$$...(1+sn\ \ 2+sn\ \ 3+sn\ ...\ (s+1)n)^{(s+1)(s+2)/2}...$$

$$...(1+(n-1)n\ \ 2+(n-1)n\ \ 3+(n-1)n\ ...\ n^2)^{n(n+1)/2},$$

$$P_3 = (1\ \ 2\ ...\ n)^1(1+n\ \ 2+n\ ...\ 2n)^2...$$

$$...(1+(n-2)n\ \ 2+(n-2)n\ ...\ (n-1)n)^{n-1}$$

$$(1+(n-1)n\ \ 2+(n-1)n\ ...\ n^2)^n,$$

$$P_4 = (1\ \ 2\ ...\ n)(1+n\ \ 2+n\ ...\ 2n)...(1+(n-1)n\ \ 2+(n-1)n\ ...\ n^2).$$

Letting $b = P_1$, $c = P_2$, $d = P_3$, $z = P_4$ and t trivial it is easy to check that these permutations satisfy the relations of $\bar{H}_n$. Consider $r + sn$ where $1 \leq r + sn \leq n^2$, $0 \leq r \leq n-1$, $0 \leq s \leq n$. We wish the permutations $b^{-1}c^{-1}bcd^{-1}$, $b^{-1}d^{-1}bdz^{-1}$, $c^{-1}d^{-1}cd$, $b^{-1}z^{-1}bz$, and $c^{-1}z^{-1}cz$ to send the element $r + sn$ to itself. So, first consider the permutation $b^{-1}c^{-1}bcd^{-1}$. Generally we have

$$r+sn \xrightarrow{b^{-1}} r+(s-1)n \xrightarrow{c^{-1}} r-s(s+1)/2+(s-1)n \xrightarrow{b}$$

$$r - s(s+1)/2 + sn \xrightarrow{c} r + s + 1 + sn \xrightarrow{d^{-1}} r + sn.$$

However, we do need to be careful here. The cases $s = 0$ and $r - s(s+1)/2 \leq 0$ must be considered separately. Consider the case $s = 0$.

$$r \xrightarrow{b^{-1}} r + (n-1)n \xrightarrow{c^{-1}} r + (n-1)n \xrightarrow{b} r \xrightarrow{c} r + 1 \xrightarrow{d^{-1}} r.$$

Now consider the case $r - s(s+1)/2 \leq 0$.

$$r + sn \xrightarrow{b^{-1}} r + (s-1)n \xrightarrow{c^{-1}} k_1 n + r - s(s+1)/2 + (s-1)n \xrightarrow{b}$$

$$k_1 n + r - s(s+1)/2 + sn \xrightarrow{c} k_2 n + r + s + 1 + sn \xrightarrow{d^{-1}} r + sn,$$

where $k_1,\ k_2$ are integers such that $1 \leq k_1 n + r - s(s+1)/2,\ k_2 n + r + s + 1 \leq n$. Now consider the permutation $b^{-1}d^{-1}bdz^{-1}$.

$$r + sn \xrightarrow{b^{-1}} r + (s-1)n \xrightarrow{d^{-1}} r - s + (s-1)n \xrightarrow{b} r - s + sn$$

$$\xrightarrow{d} r + 1 + sn \xrightarrow{z^{-1}} r + sn,$$

Again we have to be careful. The cases $s = 0$ and $r \leq s$ must be considered separately. Now consider the permutation $c^{-1}d^{-1}cd$.

$$r + sn \xrightarrow{c^{-1}} r - (s+1)(s+2)/2 + sn \xrightarrow{d^{-1}}$$

$$r-(s+1)(s+2)/2-(s+1)+sn \xrightarrow{c} r-(s+1)+sn \xrightarrow{d} r+sn,$$

Here the cases $r-(s+1)(s+2)/2 \leq 0$ and $r-(s+1)(s+2)/2-(s+1) \leq 0$, $r-(s+1)(s+2)/2 > 0$ have to be considered separately. Now consider $b^{-1}z^{-1}bz$.

$$r+sn \xrightarrow{b^{-1}} r+(s-1)n \xrightarrow{z^{-1}} r-1+(s-1)n \xrightarrow{b} r-1+sn \xrightarrow{z} r+sn,$$

Here we must look at the cases $s=0$ and $r=0,\ 1$ separately. Lastly consider the permutation $c^{-1}z^{-1}cz$.

$$r+sn \xrightarrow{c^{-1}} r-(s+1)(s+2)/2+sn \xrightarrow{z^{-1}} r-(s+1)(s+2)/2-1+sn \xrightarrow{c}$$

$$r-1+sn \xrightarrow{z} r+sn.$$

Here we have to look at the cases $r-(s+1)(s+2)/2 \leq 0$ and $r-(s+1)(s+2)/2 = 1$ separately. Since these permutations satisfy the relations of $\bar{H}_n$ they must generate a factor group of $\bar{H}_n$. Since z has order n in this factor group, it must have order n in $\bar{H}_n$. Also, since z is non-trivial in this factor group, with t trivial then we cannot have z equal to some power of t in $\bar{H}_n$. Now, since $[b,\ c]d^{-1}=1 \Leftrightarrow [c,\ b]d=1$, if we let $b=P_2$, $c=P_1$, $d^{-1}=P_3$, $t=P_4$ and z trivial we will again have permutations satisfying the relations of $\bar{H}_n$. This time the permutations generate a factor group of $\bar{H}_n$ in which t has order n and so t must have order n in $\bar{H}_n$. Also, since t is non-trivial in this factor group, with

z trivial then we cannot have t equal to some power of z in $\bar{H}_n$. Hence, in $\bar{H}_n$, we must have both z and t of order n, neither one being equal to a power of the other and H_n must have Schur multiplier of rank at least two. □

So, for $n \geq 3$ odd, G_n has trivial Schur multiplier and a subgroup of index two with multiplier of rank at least two. Therefore, by Lemma 4.1.3, G_n is inefficient. □

Section 4.2. Alternative representations of these groups

Using the ideas of Section 1.6 we can get a permutation representation for G_n by coset enumeration over a subgroup containing no non-trivial normal subgroups. We can use the following argument to show that $\langle a,\ b\rangle$ has no non-trivial normal subgroups of G_n and so coset enumeration of G_n over $\langle a,\ b\rangle$ will yield a permutation representation for G_n. Assume $1 \neq N \leq \langle a,\ b\rangle$, $N \lhd G_n$. Since $\langle a,\ b\rangle$ is dihedral of order $2n$, N contains elements of the form b^i and ab^i, $1 \leq i \leq n-1$. Now $c^{-1}bc = bd$ so $c^{-1}b^ic = (bd)^i = b^id^i$. So, if $b^i \in N$ then $b^id^i \in N$ which is a contradiction. Similarly, $ab^i \in N$ implies that $ac^2b^id^i \in N$ which also gives us a contradiction. So we have that $\langle a,\ b\rangle$ contains no non-trivial normal subgroups of G_n. Using a computer implementation of coset enumeration, [12], we can obtain coset tables for this enumeration for small values of n prime. After some permuting of the symbols in these tables, for $n = 3,\ 5,\ 7,\ 11$ permutation generators on n^2 symbols are:

$$A = (1)(2)...(n)(n+1\ \ (n-1)n+1)...(in+s\ \ (n-i)n+s)...,$$

$$0 \leq s < n,\ i \geq 1,$$

$$B = n \text{ cycles of the form} : (in+1\ \ in+2\ ...\ (i+1)n)^i,$$

$$0 \leq i \leq n-1,$$

$$C = n \text{ cycles of the form} : (i\ \ i+n\ \ i+2n\ ...\ i+(n-1)n),$$

$$D = n \text{ cycles of the form} : (in+1\ \ in+2\ ...\ (i+1)n).$$

We can check, using the same ideas as in the proof of Lemma 4.1.5, letting $a = A,\ b = B,\ c = C,\ d = D$, that these permutations satisfy the relations of G_n for larger values of n and so these permutations at least generate a factor group of these groups.

An alternative presentation for G_n, at least for $n \leq 13$, is:

$$G_n = \langle a,\ b,\ c \mid a^2 = b^n = c^n,\ abcacb = abac^{-1}bc = 1 \rangle,$$

replacing a in this presentation by the element $abcd$ in the original presentation. So, at least for $n \leq 13$ the deficiency of G_n is one. Again coset enumeration over $\langle c \rangle$ yields a permutation representation for G_n. For $n = 3,\ 5,\ 7,\ 9,\ 11$ permutation generators for G_n on $2n^2$ symbols are:

$$A = n^2 \text{ cycles of the form} : (s\ \ s+n^2),$$

$$C = \begin{cases} (in+1\ \ in+2\ ...\ (i+1)n)^{i+1},\ i \leq n-1 \\ (in+1\ \ in+2\ ...\ (i+1)n)^{2n-i},\ i \geq n. \end{cases}$$

The third generator, B, is more difficult to write down generally. It consists of n cycles of the form $(i\ \ i+n\ \ i+2n\ ...\ i+(n-1)n)$, $i \leq n$. The other n n-cycles containing the remaining n^2 symbols can be constructed as follows. Write n empty n-cycles one below the other so we can think of corresponding positions in the cycles as 'columns'. Take the symbols n at a time in order. Put the first n symbols in the first column in order. For each of the other sets of n symbols put the next set in the nearest available column to the left cyclically in order with the first symbol of the set next to the last symbol placed from the previous set. We end up with $2n^2$ mapped to n^2+1 by B. So permutation generators for G_5 are:

$$A = (1\ \ 26)(2\ \ 27)...(25\ \ 50),$$

$$B = \begin{array}{c} (1\ 6\ 11\ 16\ 21)(26\ 32\ 38\ 44\ 50)(2\ 7\ 12\ 17\ 22)(27\ 33\ 39\ 45\ 46) \\ (3\ 8\ 13\ 18\ 23)(28\ 34\ 40\ 41\ 47)(4\ 9\ 14\ 19\ 24)(29\ 35\ 36\ 42\ 48) \\ (5\ 10\ 15\ 20\ 25)(30\ 31\ 37\ 43\ 49), \end{array}$$

$$C = \begin{array}{c} (1\ 2\ 3\ 4\ 5)(31\ 35\ 34\ 33\ 32)(6\ 8\ 10\ 7\ 9)(36\ 39\ 37\ 40\ 38) \\ (11\ 14\ 12\ 15\ 13)(41\ 34\ 45\ 42\ 44)(16\ 20\ 19\ 18\ 17)(46\ 47\ 48\ 49\ 50). \end{array}$$

Section 4.3. Extensions of these inefficient groups

Lemma 4.3.1. *$G_n \times C_m$ is inefficient whenever m is odd.*

Proof. We use Theorem 1.5.4. We have:

$$M(G_n \times C_m) \cong M(G_n) \times M(C_m) \times (G_n \otimes C_m).$$

Since $G/G' = C_2$ we have $G_n \otimes C_m$ trivial and so $M(G_n \times C_m)$ is trivial. Now consider the index two subgroup of $G_n \times C_m$, $H_n \times C_m$.

$$M(H_n \times C_m) \cong M(H_n) \times M(C_m) \times (H_n \otimes C_m).$$

Since H_n has multiplier of rank at least two, $M(H_n \times C_m)$ must have rank at least two and we can use Lemma 4.1.3 to show that $G_n \times C_m$ is inefficient. □

Not all direct products involving these inefficient groups are inefficient. The direct product of G_3 with C_2 has efficient presentation:

$$\langle a,\ b,\ c \mid a^2 = b^3 = c^6,\ abcacb = abac^{-1}bc = 1\rangle.$$

We can also construct other inefficient extensions of these groups. The semidirect product of G_3 by C_3 with presentation:

$$\langle a,\ b,\ c,\ d \mid a^2 = b^3 = c^3, d^3 = abcacb = abac^{-1}bc = 1,$$

$$d^{-1}ad = b^{-1}ab,\ d^{-1}bd = c^{-1}b^{-1},\ d^{-1}cd = ab^{-1}a\rangle$$

has trivial Schur multiplier and a subgroup of index two having multiplier of rank two and so is inefficient.

Chapter 5. Some unsolved problems

Many questions on efficiency remain unanswered since, generally, it is not easy to say whether a finite group is efficient or inefficient. To show that a finite group is efficient we must find an efficient presentation for it. To show that a group is inefficient we must show that no such presentation exists. In general, neither of these problems are straightforward. In fact, finding the Schur multiplier of a finite group is difficult, even with the aid of a computer, if the group has high order.

Chapter 2 naturally raises the question of the deficiencies of the groups A_n, S_n, $n \geq 9$.

In Sections 3.2 and 3.3 of this thesis we show that for certain values of p and n, p prime, $PSL(2,\ p^n)^2$ and $PSL(2,\ p^n)^3$ are efficient. We also show that the direct product of any two or any three of these groups is efficient. It is natural to ask whether or not this is the case for all p and n, p prime. The methods used in this thesis require that the groups $PSL(2,\ p^n)$ have presentations of a particular form. For which values of p and n, p prime, does $PSL(2,\ p^n)$ have a presentation of this form?

Another natural question to ask is whether or not $PSL(2,\ p^n)^4$ is efficient and, more generally, for which values of m is $PSL(2,\ p^n)^m$ efficient. In [4] Campbell, Miyamoto, Robertson and Williams show that $PSL(2,\ 5)^m = A_5^m$ is efficient for $m \leq 4$. It is not yet known what happens to the efficiency of A_5^m as $m \to \infty$. In fact, it is not yet known whether or not the direct product of two efficient groups can be inefficient.

In [30] Wiegold asks the following question. If G is finite and non trivial, does $\text{def}(G^n) \to \infty$ as $n \to \infty$? This is obviously the case when G has non-trivial multiplier by Theorem 1.5.4. Also, if G is non-perfect, then the rank of the multiplier and hence the deficiency will tend to infinity as n tends to infinity. However, the question remains unanswered for perfect groups having trivial multiplier.

Section 4.3 gives examples illustrating the fact that the direct product of an inefficient group with an efficient group may be efficient or inefficient. However, there are no known examples of efficient direct products of two inefficient groups.

In Chapter 4 we use Lemma 4.1.3 to show that the groups in question are inefficient. This uses the fact that $\text{rank}(M(G)) \leq \text{def}(G)$ and is only useful if the groups in question have multiplier of low enough rank, and a subgroup of low enough index and high enough rank. In [29] Wamsley gives a better bound for the rank of the multiplier than the deficiency of the group. This is another group invariant and has become known as the *abelianised deficiency* of G, $\mathrm{abdef}(G)$. Wamsley shows that, for a finite group G,

$$\mathrm{rank}(M(G)) \leq \mathrm{abdef}(G) \leq \mathrm{def}(G).$$

Whether or not there exists a finite group G such that $\mathrm{abdef}(G) < \mathrm{def}(G)$ is an open problem.

In [29], Wamsley states that for any nilpotent group G, $\mathrm{abdef}(G) = \mathrm{def}(G)$. He also poses several questions relating to the efficiency of nilpotent groups. He asks whether a p-group is necessarily efficient, whether the direct product of two efficient nilpotent groups is efficient and whether any nilpotent group is inefficient. The investigation of the efficiency of p-groups naturally leads to the question of when exactly a p-group has trivial Schur multiplier. Wiegold [30] gives a good survey of the problems in this area.

In the case when G is an efficient finite group with minimum number of generators g, Wamsley [29] asks the following question. Does G necessarily have an efficient presentation on g generators?

So efficiency is an area of group theory with a lot of ground still to cover, some of the most natural questions being some of the most difficult.

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